

CHARACTERISTIC FORMS OF COMPLEX CARTAN GEOMETRIES

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ABSTRACT. We calculate relations on characteristic classes which are obstructions preventing closed Kähler manifolds from carrying holomorphic Cartan geometries. We apply these relations to give global constraints on the phase spaces of complex analytic determined and underdetermined systems of differential equations.

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1. INTRODUCTION

This article cuts away the overgrown sheaf theory from the garden of parabolic geometry. Gunning [15] and Kobayashi and Ochiai [18] p. 78 obtained relations on characteristic classes for normal projective connections following the Thomas theory of normal projective connections, instead of the Cartan theory, employing sheaves instead of principal bundles. Kobayashi and Ochiai [19] p. 207 extended

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this approach to obtain relations on characteristic classes for certain holomorphic G -structures. Cartan's theory generalizes easily to Cartan geometries, and manages abnormalcy without extra effort. We use the Cartan theory to study relations between characteristic classes.

Theorem 1. *The ring of characteristic classes of a Cartan geometry on a Kähler manifold is a quotient of the ring of characteristic forms of the model via an explicit ring morphism.*

We compute the relations on characteristics rings of various rational homogeneous varieties to give examples, and explain how to employ these results to study various types of complex analytic differential equations.

2. CARTAN GEOMETRIES

2.1. Definition.

Definition 1. A *Cartan pseudogeometry* on a manifold M , modelled on a homogeneous space G/P , is a principal right P -bundle $E \rightarrow M$, (with right P action written $r_g : E \rightarrow E$ for $g \in P$), with a 1-form $\omega \in \Omega^1(E) \otimes \mathfrak{g}$, called the *Cartan pseudoconnection* (where $\mathfrak{g}, \mathfrak{p}$ are the Lie algebras of G, P), so that ω identifies each tangent space of E with \mathfrak{g} . For each $A \in \mathfrak{g}$, let \vec{A} be the vector field on E satisfying $\vec{A} \lrcorner \omega = A$. A Cartan pseudogeometry is called a *Cartan geometry* (and its Cartan pseudoconnection called a *Cartan connection*) if (1) the vectors fields \vec{A} generate the infinitesimal right action:

$$\vec{A} = \left. \frac{d}{dt} r_{e^{tA}} \right|_{t=0}$$

for all $A \in \mathfrak{p}$, and (2) the Cartan connection transforms in the adjoint representation: $r_g^* \omega = \text{Ad}_g^{-1} \omega$ for all $g \in P$.

If all of the manifolds, Lie groups, differential forms, and maps above are complex analytic, we will say that M bears a *complex* Cartan geometry. All Cartan geometries will henceforth be assumed complex.

In most of our examples, G will be a reductive algebraic group and P will be a parabolic subgroup.

2.2. Examples.

Example 1 (Affine connections). Let M be a complex manifold with an affine connection (i.e. a connection on TM). Lets see why this is a Cartan geometry modelled on affine space. Let G be the group of affine transformations of \mathbb{A}^n , and $P = \text{GL}(n, \mathbb{C})$ the linear transformations (i.e. fixing a point). So $G/P = \mathbb{A}^n$. Let n be the dimension of M . Let $\pi : E \rightarrow M$ be the principal right $\text{GL}(n, \mathbb{C})$ -bundle associated to the tangent bundle. So a point e of E is a basis $e = (e_1, e_2, \dots, e_n)$ for $T_m M$ for some point m of M . Let ω_0^i be the 1-forms on E defined by

$$v \lrcorner \omega_0^i e_i = \pi'(e) \cdot v.$$

Let (ω_j^i) be the connection 1-form; $(\omega_j^i) \in \Omega^1(E) \otimes \mathfrak{gl}(n, \mathbb{C})$.

Let

$$\omega = \begin{pmatrix} 0 & 0 \\ \omega_0^i & \omega_j^i \end{pmatrix} \in \Omega^1(E) \otimes \mathfrak{g}.$$

In this way, affine connections (even with torsion) are Cartan geometries modelled on affine space.

Example 2 (Projective connections). Write the standard basis of \mathbb{C}^{n+1} as e_0, e_1, \dots, e_n . Let $G = \mathrm{SL}(n+1, \mathbb{C})$, and P be the stabilizer of the line $\mathbb{C}e_0$. Then $G/P = \mathbb{P}^n$. A Cartan geometry modelled on \mathbb{P}^n is called a *projective connection* (see Cartan [12]).

Example 3 (Conformal connections). Let $G = \mathrm{SO}(n+2, \mathbb{C})$, and P be the stabilizer of a null line in \mathbb{C}^{n+2} . The quotient $Q = G/P$ is the hyperquadric (the set of null lines in \mathbb{C}^{n+2}). A Cartan geometry modelled on the hyperquadric is called a *conformal connection*.

It is well known (Cartan [9]) that a conformal structure on manifold of at least three dimensions determines and is determined by a unique conformal connection. Not every conformal connection occurs this way; those which do are called *normal*. (See Čap [27] for the definition of normalcy.)

Example 4 (2nd order systems of ordinary differential equations). Tanaka [26] showed that every second order system of ordinary differential equations

$$\frac{d^2 y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right),$$

with $x \in \text{open} \subset \mathbb{C}$ and $y \in \text{open} \subset \mathbb{C}^n$, determines a unique Cartan geometry on the “phase space” of points $(x, y, p) \in \text{open} \subset \mathbb{C}^{2n+1}$, (where p formally represents $\frac{dy}{dx}$), modelled on $\mathbb{P}T\mathbb{P}^{n+1}$. In the model, we think of (x, y) as coordinates of an affine chart on \mathbb{P}^{n+1} . In those coordinates, $\mathbb{P}T\mathbb{P}^{n+1}$ has points (x, y, ξ) with $\xi \in \mathbb{P}T_{(x,y)}\mathbb{C}^{n+1} = \mathbb{P}^n$. Take p as an affine chart on \mathbb{P}^n , so that ξ is the span of $(1, p) \in \mathbb{C}^{n+1}$, i.e. in homogeneous coordinates $\xi = [1 : p]$. The 2nd order system of differential equations in the model is $dy = p dx, dp = 0$.

Example 5 (2-plane fields on 5-folds). It is well known (see Bryant [7], Cartan [8], Gardner [14], Nurowski [21], Sternberg [25], Tanaka [26]) that any 5-dimensional manifold equipped with a “nondegenerate” 2-plane field bears a canonical Cartan geometry modelled on the hyperquadric $Q^5 = G_2/P$ where P is the stabilizer of a null line in the (unique up to scalar) G_2 -invariant quadratic form on the (unique up to isomorphism) irreducible G_2 -module \mathbb{C}^7 .

Example 6 (3rd order ordinary differential equations). Sato & Yoshikawa [22] show that any 3rd order ordinary differential equation

$$\frac{d^3 y}{dx^3} = f\left(x, y, \frac{dy}{dx}, \frac{d^2 y}{dx^2}\right)$$

determines on its (4-dimensional) phase space a Cartan geometry modelled on $\mathrm{SO}(5, \mathbb{C})/B$, with B the Borel subgroup. The Cartan geometry is invariant under local contact isomorphisms of 3rd order ODEs.

2.3. Curvature. Take a Cartan geometry $E \rightarrow M$ with Cartan connection ω , modelled on G/P . The *curvature* is

$$\begin{aligned} \nabla\omega &= d\omega + \frac{1}{2}[\omega, \omega] \\ &= \frac{1}{2}\kappa\omega^{G/P} \wedge \omega^{G/P}, \end{aligned}$$

where $\omega^{G/P} = \omega \pmod{\mathfrak{p}} \in \Omega^1(E) \otimes (\mathfrak{g}/\mathfrak{p})$ is the soldering form and $\kappa : E \rightarrow \mathfrak{g} \otimes \Lambda^2(\mathfrak{g}/\mathfrak{p})^*$. Under right P -action $r_g : E \rightarrow E$,

$$r_g^* \omega = \text{Ad}_g^{-1} \omega, r_g^* \nabla \omega = \text{Ad}_g^{-1} \nabla \omega.$$

Example 7 (Affine connections). For an affine connection,

$$d\omega + \frac{1}{2} [\omega, \omega] = \begin{pmatrix} 0 & 0 \\ \kappa_{k\ell}^i & \kappa_{jk\ell}^i \end{pmatrix} \omega^k \wedge \omega^\ell.$$

The quantity $\kappa_{k\ell}^i$ is the torsion of the affine connection, while $\kappa_{jk\ell}^i$ is the curvature, in the coframe on $T_m M$ which pulls back to ω^i on $T_e E$.

3. WHY WE WOULD EXPECT RELATIONS ON CHARACTERISTIC CLASSES

Clearly the G -invariant vector bundles (or coherent sheaves) on any homogeneous space G/P are precisely the vector bundles $G \times_P W$, where W can be any P -module. (For many homogeneous spaces, including all compact complex homogeneous spaces, all vector bundles are G -invariant.) Suppose that $E \rightarrow M$ is a Cartan geometry modelled on G/P . To each such vector bundle, we can associate a vector bundle $E \times_P W \rightarrow M$ on every Cartan geometry modelled on G/P . This mapping

$$(\text{vector bundles on } G/P)^G \rightarrow \text{vector bundles on } M$$

takes sums to sums, subspaces to subspaces, quotients to quotients, exact sequences to exact sequences, tensor products to tensor products, etc. Moreover, it takes $G \times_P (\mathfrak{g}/\mathfrak{p})$ to TM , so strikes all tensor bundles on M . Clearly we expect to see a map on the characteristic class ring, or at least the part of the characteristic class ring arising from G -invariant vector bundles on G/P .

4. NOTATION FOR VARIOUS SUBGROUPS

We will henceforth fix a maximal complex reductive subgroup $LA \subset P$, and assume that we have fixed a direct sum decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{a} \oplus \mathfrak{n}$, of LA -modules, with $\mathfrak{l} \oplus \mathfrak{a} \subset \mathfrak{p}$ the Lie subalgebra of LA , and $\mathfrak{l}, \mathfrak{a}, \mathfrak{n} \subset \mathfrak{p}$ semisimple, abelian and nilpotent complex subalgebras. Also assume that we have fixed a direct sum decomposition $\mathfrak{g} = \mathfrak{p} + \mathfrak{g}/\mathfrak{p}$ of LA -modules. We won't need to assume that there is a closed subgroup $N \subset P$ with Lie algebra \mathfrak{n} , nor that there are closed subgroups L, A with Lie algebras $\mathfrak{l}, \mathfrak{a}$.

We will write H^c for a maximal compact subgroup of any group H , and $\mathfrak{h}^c \subset \mathfrak{h}$ for the Lie algebra of $H^c \subset H$. We will assume that we have picked maximal compact subgroups of P and LA so that $(LA)^c = LA \cap P^c$.

4.1. Reductions of structure group. If $E \rightarrow M$ is any Cartan geometry, then there is a smooth (not necessarily holomorphic) reduction of structure group to a principal right P^c -bundle which we will call $E^c \rightarrow M$ (see Steenrod [24] 12.14). Moreover, $E = E^c \times_{P^c} P$. We will also consider the principal right LA -bundle $E^{LA} = E^c \times_{P^c} LA$. Obviously there are many choices of reduction of structure group $E^c \subset E$, but any choice will lead us to the same characteristic classes.

Example 8 (Projective connections). For a projective connection, we can write the elements of \mathfrak{g} as matrices, say

$$\omega = \begin{pmatrix} \omega_0^0 & \omega_j^0 \\ \omega_0^i & \omega_j^i \end{pmatrix}$$

with $i, j = 1, \dots, n$, and $\omega_0^0 + \omega_i^i = 0$. Elements of the subalgebra $\mathfrak{p} \subset \mathfrak{g}$ look like

$$\begin{pmatrix} \omega_0^0 & \omega_j^0 \\ 0 & \omega_j^i \end{pmatrix}.$$

It is helpful to split up the Lie algebra into a sum:

$$\mathfrak{g} = \mathfrak{g}/\mathfrak{p} \oplus \mathfrak{l} \oplus \mathfrak{a} \oplus \mathfrak{n},$$

(*not* a sum of subalgebras, but only of vector subspaces) where the various parts are given by splitting matrices as

$$\omega = \begin{pmatrix} 0 & 0 \\ \omega_0^i & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \omega_j^i - \frac{\delta_j^i}{n} \omega_k^k \end{pmatrix} + \begin{pmatrix} \omega_0^0 & 0 \\ 0 & \frac{\delta_j^i}{n} \omega_k^k \end{pmatrix} + \begin{pmatrix} 0 & \omega_j^0 \\ 0 & 0 \end{pmatrix}.$$

This corresponds on the group G to splitting up any matrix $g \in G$, say

$$g = \begin{pmatrix} g_0^0 & g_j^0 \\ g_0^i & g_j^i \end{pmatrix}$$

as

$$g = \begin{pmatrix} 1 & g_m^0 h_j^m \\ 0 & \delta_j^i \end{pmatrix} \begin{pmatrix} g_0^0 - g_m^0 h_p^m g_0^p & 0 \\ 0 & g_j^i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ h_q^k g_0^q & \delta_\ell^k \end{pmatrix}$$

(essentially the Harish-Chandra decomposition), where here the components are split up in the reverse order, and the $\mathfrak{l} \oplus \mathfrak{a}$ parts are combined.

We can write the Cartan connection 1-form ω on any projective connection $E \rightarrow M$ in the same manner. It is helpful to split up the Maurer–Cartan 1-form into pieces as

$$\begin{aligned} \omega^{G/P} &= (\omega_0^i) \in \Omega^1(G) \otimes (\mathfrak{g}/\mathfrak{p}) \\ \omega^{LA} &= (\omega_j^i + \delta_j^i \omega_k^k) \in \Omega^1(G) \otimes (\mathfrak{l} \oplus \mathfrak{a}) \\ \omega^N &= (\omega_j^0) \in \Omega^1(G) \otimes \mathfrak{n}. \end{aligned}$$

The 1-forms $\omega^{G/P}$ are semibasic, i.e. they vanish on the fibers of $E \rightarrow M$.

We write ω^i for $\omega^{G/P} = \omega_0^i$, ω_i for $\omega^N = \omega_0^i$, and write ω^{LA} as γ_j^i . The curvature splits up then as in table 1 on the following page. Looking at the equations for γ_j^i , i.e. for ω^{LA} , we can define

$$\nabla^{LA} \omega^{LA} = (d\gamma_j^i + \gamma_k^i \wedge \gamma_j^k),$$

and find immediately that

$$\nabla^{LA} \omega^{LA} = \left(-(\omega_j \delta_k^i + \omega_k \delta_j^i) \wedge \omega^k + \frac{1}{2} K_{jkl}^i \omega^k \wedge \omega^l \right)$$

It looks at first sight as if ω^{LA} is a connection 1-form. But its “curvature” $\nabla^{LA} \omega^{LA}$ is *not* semibasic, since it involves ω_i terms. So ∇^{LA} is only vaguely reminiscent of a connection. Nevertheless, we will see that

$$-(\omega_j \delta_k^i + \omega_k \delta_j^i) \wedge \omega^k$$

behaves very much like a curvature term, and controls characteristic classes.

The subgroup $G^c = \mathrm{SU}(n+1)$ acts on \mathbb{P}^n transitively. On $\mathrm{SU}(n+1)$, ω lives in $\mathfrak{su}(n+1)$, so that $\omega^{LA} \in \mathfrak{u}(n)$, and $\omega^N = -(\omega^{G/P})^*$, i.e. $\omega_i = -\omega^{\bar{i}}$. Therefore

$$\begin{aligned}
\nabla\omega^i &= d\omega^i + \gamma_j^i \wedge \omega^j \\
&= \frac{1}{2}K_{kl}^i \omega^k \wedge \omega^l \\
\nabla\gamma_j^i &= d\gamma_j^i + \gamma_k^i \wedge \gamma_j^k - (\omega_j \delta_k^i + \omega_k \delta_j^i) \wedge \omega^k \\
&= \frac{1}{2}K_{jkl}^i \omega^k \wedge \omega^l \\
\nabla\omega_i &= d\omega_i - \gamma_i^j \wedge \omega_j \\
&= \frac{1}{2}K_{ikl} \omega^k \wedge \omega^l \\
0 &= K_{jk}^i + K_{kj}^i \\
0 &= K_{jkl}^i + K_{jlk}^i \\
0 &= K_{ikl} + K_{ilk}.
\end{aligned}$$

TABLE 1. The structure equations of a projective connection

when we restrict from G to G^c , ω^{LA} becomes a connection, and its curvature is precisely

$$\begin{aligned}
\nabla^{LA}\omega^{LA} &= -(\omega_j \delta_k^i + \omega_k \delta_j^i) \wedge \omega^k \\
&= (\omega^{\bar{j}} \delta_k^i + \omega^{\bar{k}} \delta_j^i) \wedge \omega^k,
\end{aligned}$$

the curvature of the Fubini–Study metric.

If we pick any reduction $E^c \subset E$ on any projective connection, we find

$$\nabla^{LA}\omega^{LA} = (\omega_j \delta_k^i + \omega_k \delta_j^i) \wedge \omega^k + \frac{1}{2}\kappa_{jk\ell}^i \omega^k \wedge \omega^\ell.$$

The last term,

$$\frac{1}{2}\kappa_{jk\ell}^i \omega^k \wedge \omega^\ell,$$

is a semibasic $(2,0)$ -form. Therefore the $(1,1)$ -terms of the curvature appear only in the terms with no $\kappa_{jk\ell}^i$ factor. The curvature of the connection ω^{LA} on the reduction has $(1,1)$ -part looking very much like the Fubini–Study metric, independent of the curvature of the Cartan connection.

Keep in mind that in a general Cartan geometry, we don't have any control on the actual ω_i terms in the reduction. So the $(1,1)$ -terms that arise might not be given by $\omega_i = -\omega^{\bar{i}}$, but instead by some more complicated expression. However, the main point is that the $(1,1)$ -terms are not (directly) influenced by the curvature κ of the Cartan geometry at each point.

Example 9 (2^{nd} order systems of ordinary differential equations). We immediately see the similarity to projective connections: write the Cartan connection 1-form

$$\omega = \begin{pmatrix} \omega_0^0 & \omega_1^0 & \omega_j^0 \\ \omega_0^1 & \omega_1^1 & \omega_j^1 \\ \omega_0^i & \omega_1^i & \omega_j^i \end{pmatrix}$$

for $i, j = 2, \dots, n+1$. Then the stabilizer subgroup $P \subset G = \mathbb{P}\mathrm{SL}(n+1, \mathbb{C})$ of a point of $\mathbb{P}T\mathbb{P}^{n+1}$ has Lie algebra the image of

$$\begin{pmatrix} \omega_0^0 & \omega_1^0 & \omega_j^0 \\ 0 & \omega_1^1 & \omega_j^1 \\ 0 & 0 & \omega_j^i \end{pmatrix}.$$

So we write

$$\begin{aligned} \omega^{G/P} &= (\omega_0^1, \omega_0^i, \omega_1^i) \\ \omega^{LA} &= (\omega_0^0, \omega_1^1, \omega_j^i) \\ \omega^N &= (\omega_1^0, \omega_j^0, \omega_j^1). \end{aligned}$$

The structure equations of the model give

$$\begin{aligned} d\omega_0^0 &= -\omega_1^0 \wedge \omega_0^1 - \omega_k^0 \wedge \omega_0^k \\ d\omega_1^1 &= -\omega_0^1 \wedge \omega_1^0 - \omega_k^1 \wedge \omega_1^k \\ d\omega_j^i &= -\omega_0^i \wedge \omega_j^0 - \omega_1^i \wedge \omega_j^1 - \omega_k^i \wedge \omega_j^k. \end{aligned}$$

On the maximal compact subgroup $G^c = \mathrm{SU}(n+1) \subset G = \mathrm{SL}(n+1, \mathbb{C})$, we find

$$\omega^N = -(\omega^{G/P})^*,$$

so once again we find that

$$\begin{aligned} \nabla^{LA} \omega^{LA} &= \begin{pmatrix} d\omega_0^0 \\ d\omega_1^1 \\ d\omega_j^i + \omega_k^i \wedge \omega_j^k \end{pmatrix} \\ &= \begin{pmatrix} \omega_0^{\bar{1}} \wedge \omega_0^1 + \omega_0^{\bar{k}} \wedge \omega_0^k \\ \omega_1^{\bar{0}} \wedge \omega_1^0 + \omega_1^{\bar{k}} \wedge \omega_1^k \\ \omega_j^{\bar{i}} \wedge \omega_j^i + \omega_j^{\bar{1}} \wedge \omega_j^1 \end{pmatrix} \end{aligned}$$

is exactly the curvature of the standard $\mathrm{SU}(n+1)$ -invariant Kähler metric on $\mathbb{P}T\mathbb{P}^{n+1}$.

Any Cartan geometry with the same model will have the same structure equations, except for Cartan connection curvature terms. These terms are $(2, 0)$ -terms expressed in terms of the semibasic $(1, 0)$ -forms, i.e. components of $\omega^{G/P}$, so linear combinations of

$$\omega_0^1 \wedge \omega_0^i, \omega_0^1 \wedge \omega_1^i, \omega_0^i \wedge \omega_0^j, \omega_0^i \wedge \omega_1^j, \omega_1^i \wedge \omega_1^j.$$

So on any reduction $E^{LA} \subset E$, we will find

$$\nabla^{LA} \omega^{LA} + \begin{pmatrix} \omega_1^0 \wedge \omega_0^1 - \omega_k^0 \wedge \omega_0^k \\ \omega_0^1 \wedge \omega_1^0 - \omega_k^1 \wedge \omega_1^k \\ \omega_0^i \wedge \omega_j^0 - \omega_1^i \wedge \omega_j^1 \end{pmatrix}$$

is $(2, 0)$, a complex linear multiple of the semibasic $(2, 0)$ -forms. Therefore the Cartan connection curvature terms never influence the $(1, 1)$ part of the curvature of the reduction.

Example 10 (2-plane fields on 5-folds). On \mathbb{C}^7 , consider the 3-form

$$\phi = dx^{567} + dx^{125} - dx^{345} + dx^{136} + dx^{246} + dx^{147} - dx^{237}$$

where dx^1, \dots, dx^7 is a basis of \mathbb{C}^{7*} and $dx^{ij} = dx^i \wedge dx^j$ etc. We will always use this basis of \mathbb{C}^{7*} , and in so doing we are following McLean [20]. Following Bryant [6] we know that the group of linear transformations of \mathbb{C}^7 fixing ϕ is G_2 . Moreover, the subgroup preserving the real locus $\mathbb{R}^7 \subset \mathbb{C}^7$ is precisely the compact form of G_2 , which we will call G_2^c . For the moment, let's work with the compact form, but of course all of the equations are identical when complexified.

The group G_2^c preserves the usual Euclidean quadratic form

$$\sum (dx^i)^2.$$

Write \mathbb{H} for the quaternions, L_q for left multiplication by q and R_q for right multiplication by q . For exterior forms α and β valued in the quaternions (or in any associative algebra), of degrees a and b respectively:

$$\begin{aligned} L_\alpha \wedge \beta &= \alpha \wedge \beta \\ R_\alpha \wedge \beta &= (-1)^{ab} \beta \wedge \alpha \\ L_\alpha \wedge L_\beta &= L_{\alpha \wedge \beta} \\ L_\alpha \wedge R_\beta &= (-1)^{ab} R_\beta \wedge L_\alpha \\ R_\alpha \wedge R_\beta &= (-1)^{ab} R_{\beta \wedge \alpha}. \end{aligned}$$

As McLean [20] explains, the Lie algebra of G_2^c can be written as

$$\omega = \begin{pmatrix} L_\lambda - R_\rho & \beta \\ -{}^t\beta & L_\rho - R_\rho \end{pmatrix}$$

in its \mathbb{R}^7 representation. The meaning of ${}^t\beta$ is that we take β , valued in $\text{Im } \mathbb{H}^* \otimes \mathbb{H}$ and use the invariant metric $q \mapsto q\bar{q}$ on \mathbb{H} to identify $\text{Im } \mathbb{H}^* \otimes \mathbb{H}$ with $\text{Im } \mathbb{H} \otimes \mathbb{H}^*$. Moreover, McLean points out that β satisfies

$$\beta_1 i + \beta_2 j + \beta_3 k = 0.^1$$

To put it more concretely (and make explicit what ${}^t\beta$ means), the Maurer–Cartan 1-form is

$$\omega = \begin{pmatrix} 0 & -\lambda^1 + \rho^1 & -\lambda^2 + \rho^2 & -\lambda^3 + \rho^3 & \beta_1^0 & \beta_2^0 & \beta_3^0 \\ \lambda^1 - \rho^1 & 0 & -\lambda^3 - \rho^3 & \lambda^2 + \rho^2 & \beta_1^1 & \beta_2^1 & \beta_3^1 \\ \lambda^2 - \rho^2 & \lambda^3 + \rho^3 & 0 & -\lambda^1 - \rho^1 & \beta_1^2 & \beta_2^2 & \beta_3^2 \\ \lambda^3 - \rho^3 & -\lambda^2 - \rho^2 & \lambda^1 + \rho^1 & 0 & \beta_1^3 & \beta_2^3 & \beta_3^3 \\ -\beta_1^0 & -\beta_1^1 & -\beta_1^2 & -\beta_1^3 & 0 & -2\rho^3 & 2\rho^2 \\ -\beta_2^0 & -\beta_2^1 & -\beta_2^2 & -\beta_2^3 & 2\rho^3 & 0 & -2\rho^1 \\ -\beta_3^0 & -\beta_3^1 & -\beta_3^2 & -\beta_3^3 & -2\rho^2 & 2\rho^1 & 0 \end{pmatrix}$$

with

$$\begin{pmatrix} \beta_1^1 + \beta_2^2 + \beta_3^3 \\ \beta_1^0 + \beta_2^3 - \beta_3^2 \\ -\beta_1^3 + \beta_2^0 + \beta_3^1 \\ \beta_1^2 - \beta_2^1 + \beta_3^0 \end{pmatrix} = 0.$$

¹McLean actually says that $i\beta_1 + j\beta_2 + k\beta_3 = 0$, which turns out not to be true. It does not affect his results.

The Maurer–Cartan structure equation $d\omega = -\frac{1}{2}[\omega, \omega]$ expands out to

$$\begin{aligned} d\lambda + \lambda \wedge \lambda &= (\beta \wedge {}^t\beta)_+ \\ d\rho + \rho \wedge \rho &= -(\beta \wedge {}^t\beta)_- \\ d\beta &= -(L_\lambda - R_\rho) \wedge \beta - \beta \wedge (L_\rho - R_\rho) \end{aligned}$$

where, since $\beta \wedge {}^t\beta$ is a 2-form valued in $\mathfrak{so}(4) = \mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_-$, we can split it into

$$\beta \wedge {}^t\beta = (\beta \wedge {}^t\beta)_+ + (\beta \wedge {}^t\beta)_-.$$

The line spanned by $e_6 + \sqrt{-1}e_7$ in \mathbb{C}^7 is a null line for the quadratic form. The subgroup of G_2^c preserving the 2-plane $e_6 \wedge e_7$ is $P^c = \mathrm{Sp}(1) \times \mathrm{U}(1) / \pm 1$, and its Lie algebra is cut out by setting $\beta = 0$ and $\rho^2 = \rho^3 = 0$:

$$\begin{pmatrix} L_\lambda - R_{i\rho^1} & 0 \\ 0 & L_{i\rho^1} - R_{i\rho^1} \end{pmatrix}.$$

Therefore on G_2^c ,

$$\begin{aligned} (1) \quad \nabla^{LA} \omega^{LA} &= \nabla^{LA} \begin{pmatrix} \lambda \\ \rho^1 \end{pmatrix} \\ (2) \quad &= \begin{pmatrix} (\beta \wedge {}^t\beta)_+ \\ -2\rho^2 \wedge \rho^3 - \frac{1}{2}(\beta_2^a \wedge \beta_3^a) \end{pmatrix}. \end{aligned}$$

Again to be more specific,

$$\begin{aligned} (\beta \wedge {}^t\beta)_+ &= \frac{i}{2} (-\beta_a^0 \wedge \beta_a^1 + \beta_a^3 \wedge \beta_a^2) \\ &\quad + \frac{j}{2} (-\beta_a^0 \wedge \beta_a^2 + \beta_a^1 \wedge \beta_a^3) \\ &\quad + \frac{k}{2} (-\beta_a^0 \wedge \beta_a^3 + \beta_a^2 \wedge \beta_a^1). \end{aligned}$$

All of these equations hold in the complex form of G_2 as well. The complex subgroup P fixing the null line through $e_6 + \sqrt{-1}e_7$ has Lie algebra given by

$$\begin{aligned} \beta_2 + \beta_3 \sqrt{-1} &= 0, \\ \rho^2 + \rho^3 \sqrt{-1} &= 0. \end{aligned}$$

Any nondegenerate holomorphic 2-plane field on a complex 5-manifold gives rise to a Cartan geometry modelled on G_2/P , which has structure equations as above but with curvature “correction terms”:

$$d\omega + \omega \wedge \omega = \kappa \omega^{G/P} \wedge \omega^{G/P}$$

and

$$\omega^{G/P} = \begin{pmatrix} \beta_2 + \beta_3 \sqrt{-1} \\ \rho^2 + \rho^3 \sqrt{-1} \end{pmatrix}.$$

Again, the (1,1)-terms in $\nabla^{LA} \omega^{LA}$ are identical to the expressions of equation 2 (the model), while the Cartan connection affects only the (2,0)-terms:

$$\nabla^{LA} \omega^{LA} = \begin{pmatrix} (\beta \wedge {}^t\beta)_+ \\ -2\rho^2 \wedge \rho^3 - \frac{1}{2}(\beta_2^a \wedge \beta_3^a) \end{pmatrix} + (2,0) \text{ curvature terms.}$$

5. CHARACTERISTIC FORMS

Take $E \rightarrow M$ any Cartan geometry, and pick a reduction $E^{LA} \subset E$. On E^{LA} , the Cartan connection ω splits into

$$\omega = \omega^{G/P} \oplus \omega^{LA} \oplus \omega^N \in \Omega^1(E^{LA}) \otimes (\mathfrak{g}/\mathfrak{p} \oplus \mathfrak{l} \oplus \mathfrak{a} \oplus \mathfrak{n}).$$

These 1-forms vary under right LA -action as the obvious LA -modules.

Lemma 1. *On any reduction of structure group $E^{LA} \subset E$, $\omega^N = t\omega^{G/P}$, for some $t : E^{LA} \rightarrow (\mathfrak{g}/\mathfrak{p})^* \otimes_{\mathbb{R}} \mathfrak{n}$.*

Remark 1. This t is not holomorphic for the generic choice of reduction E^{LA} .

Proof. On $E^{LA} \subset E$, we have $\vec{A} \lrcorner \omega = A$, for $A \in \mathfrak{l} \oplus \mathfrak{a}$. So splitting

$$\omega = \omega^{G/P} \oplus \omega^{LA} \oplus \omega^N,$$

we have $\vec{A} \lrcorner \omega = \vec{A} \lrcorner \omega^{LA}$, for $A \in \mathfrak{l} \oplus \mathfrak{a}$. Therefore $\vec{A} \lrcorner \omega^N = 0$, so ω^N is semibasic. Let $\pi : E \rightarrow M$ be the bundle map. Clearly $\omega^{G/P} = \omega + \mathfrak{p}$ yields isomorphisms

$$(\pi^*TM)_e \longrightarrow T_e E / \ker \pi'(e) \longrightarrow \mathfrak{g}/\mathfrak{p}.$$

So every semibasic 1-form is a multiple of $\omega^{G/P}$. \square

Remark 2. The 1-form ω^{LA} is a connection for the bundle $E^{LA} \rightarrow M$ (not a Cartan connection, unless M has dimension 0).

We can calculate the curvature of this connection as

$$\begin{aligned} \nabla^{LA} \omega^{LA} &= d\omega^{LA} + \frac{1}{2} [\omega^{LA}, \omega^{LA}] \\ &= K^{LA} \omega^{G/P} \wedge \omega^{G/P}. \end{aligned}$$

The characteristic forms of E^{LA} are the differential forms

$$P(\nabla^{LA} \omega^{LA}) \in \Omega^{\text{even}}(M),$$

computed from complex polynomials P on the Lie algebra which are invariant under LA . We will write this form as $P(E^{LA})$.

We may also consider the same Lie algebra splitting applied to ω on E . This splitting will not be invariant under the structure group P , but only under LA . Nonetheless, we can write down differential forms

$$P(E) = P\left(d\omega^{LA} + \frac{1}{2} [\omega^{LA}, \omega^{LA}]\right) \in \Omega^{\text{even}}(E).$$

They may refuse to descend to M , being only invariant under LA , but in general (as we will see in examples) not under \mathfrak{n} .

Obviously, the inclusion $E^{LA} \subset E$ pulls back $P(E)$ to $P(E^{LA})$. It is in this sense that we will think of $P(E)$ as a characteristic form of E , even though it is not defined downstairs on M .

Lemma 2. *$P(E)$ is closed.*

Proof. The proof is as for the usual theory of characteristic forms. Because P is LA -invariant,

$$P(A) = P(\text{Ad}_g A),$$

for any $g \in LA$. When we polarize P , this gives

$$0 = P(A, A, \dots, [B, A], A, \dots, A)$$

for any $A, B \in \mathfrak{l} \oplus \mathfrak{a}$. By the Jacobi identity for Lie algebras,

$$[\omega^{LA}, [\omega^{LA}, \omega^{LA}]] = 0.$$

We will use the expression $\nabla^{LA}\omega^{LA}$ to mean

$$\nabla^{LA}\omega^{LA} = d\omega^{LA} + \frac{1}{2}[\omega^{LA}, \omega^{LA}]$$

even on the bundle E , where this expression cannot be interpreted as a covariant derivative in any sense. Calculate

$$\begin{aligned} dP(E) &= d(P(\nabla^{LA}\omega^{LA})) \\ &= d(P(\nabla^{LA}\omega^{LA}, \dots, \nabla^{LA}\omega^{LA})) \\ &= \sum P(\nabla^{LA}\omega^{LA}, \dots, d\nabla^{LA}\omega^{LA}, \dots, \nabla^{LA}\omega^{LA}) \\ &= \sum P\left(\nabla^{LA}\omega^{LA}, \dots, d\left(d\omega^{LA} + \frac{1}{2}[\omega^{LA}, \omega^{LA}]\right), \dots, \nabla^{LA}\omega^{LA}\right) \\ &= \sum P(\nabla^{LA}\omega^{LA}, \dots, [d\omega^{LA}, \omega^{LA}], \dots, \nabla^{LA}\omega^{LA}) \\ &= \sum P(\nabla^{LA}\omega^{LA}, \dots, [\nabla^{LA}\omega^{LA}, \omega^{LA}], \dots, \nabla^{LA}\omega^{LA}) \end{aligned}$$

which vanishes by the invariance of P . \square

Lemma 3. *The cohomology class of $P(E^{LA})$ is independent of the choice of reduction $E^{LA} \subset E$.*

Proof. Because LA contains the maximal compact subgroup of P , P/LA is contractible. Therefore the bundle $E/LA \rightarrow M$ (whose smooth sections are smooth [not necessarily holomorphic] reductions of structure group) has contractible fibers, and so any two smooth sections are smoothly homotopic through smooth sections. The cohomology class is the pullback of the cohomology class of $P(E)$. \square

Obviously characteristic forms and classes of Cartan geometries pull back under local isomorphisms.

Lemma 4. *For any choice of reduction of structure group $E^{LA} \subset E$ (from P to LA), the principal right LA -bundle $E^{LA} \rightarrow M$ is a holomorphic principal bundle, although perhaps not holomorphically embedded inside E .*

Proof. The $(1,0)$ -forms ω^N on E will not generally pull back in any simple way to E^{LA} . However, $\omega^{G/P}$ and ω^{LA} will pull back to a complex coframing on E^{LA} . Therefore there is a unique almost complex structure on E^{LA} for which these forms are $(1,0)$:

$$\begin{pmatrix} Jv \lrcorner \omega^{G/P} \\ Jv \lrcorner \omega^{LA} \end{pmatrix} = i \begin{pmatrix} v \lrcorner \omega^{G/P} \\ v \lrcorner \omega^{LA} \end{pmatrix}.$$

Lets look back at E . Because $\mathfrak{n} \subset \mathfrak{g}$ is a Lie subalgebra, $[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{n}$, so there are no $\omega^N \wedge \omega^N$ terms in $d\omega^{G/P}$ or in $d\omega^{LA}$. Therefore pulling back to E^{LA} , even though ω^N might have become $(1,0) + (0,1)$, there are no $(0,2)$ -terms in $d\omega^{G/P}$ or in $d\omega^{LA}$. So the almost complex structure is a complex structure.

On E , the $(1,0)$ -form $\omega^{G/P}$ is semibasic, and at each point of E it is pulled back from a $(1,0)$ -form on M which forms a complex coframing. By pullback, the same is true on E^{LA} . Therefore $E^{LA} \rightarrow M$ is a holomorphic map.

The group action of LA on E^{LA} transforms $\omega^{G/P}$ and ω^{LA} in the obvious representation, which is complex linear, and therefore is complex analytic. \square

Remark 3. The apparent miracle that a smooth submanifold $E^{LA} \subset E$ should inherit a complex structure from E without being a complex submanifold, is not really so remarkable. If there is a closed connected complex subgroup $N \subset P$ with Lie algebra $\mathfrak{n} \subset \mathfrak{p}$, then the composition $E^{LA} \rightarrow E \rightarrow E/N$ is a biholomorphism.

The complex vector space splitting

$$\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

ensures that at least locally we can find a holomorphic “local subgroup” transverse to LA in P . Therefore we can locally holomorphically construct a transversal to the LA -action on E . This transversal replaces E/N , and then we see that the complex structure on E^{LA} varies biholomorphically with the choice of reduction.

Remark 4. The characteristic form $P(E)$ is a holomorphic $(p, 0)$ -form ($p = 2 \deg P$), because the entire construction on E is complex analytic. However, $P(E^{LA})$ is a sum of forms of various degrees (p, q) with $p + q = 2 \deg P$. We have little control on the degrees which can occur. But the summands in degrees other than (p, p) will usually vanish in cohomology.

Proposition 1 (Singer [23]). *The characteristic classes of a holomorphic principal bundle on a Kähler manifold are (p, p) -classes.*

Proof. First we need:

Lemma 5 (Singer [23]). *There is a unique connection on E^c whose induced connection on E^{LA} is compatible with the complex structure, i.e. so that $(\nabla^{LA})^{(0,1)} = \bar{\partial}$.*

Proof. Consider a holomorphic local trivialization of E^{LA} and a smooth local trivialization of E^c , defined in the same open set $U \subset M$. They are related by a map

$$(z, g) \mapsto (z, k(z)g),$$

for $g \in (LA)^c$, and $k : U \rightarrow LA$ is smooth. Split diffeomorphically by Cartan decomposition $LA = (LA)^c e^\Pi$, where $\Pi \subset \mathfrak{l} \oplus \mathfrak{a}$ is the locus fixed under a Cartan involution. Changing the smooth trivialization, we can replace $k(z)$ by $h(z)k(z)$ for any $h : U \rightarrow (LA)^c$. Therefore we can arrange that $k : U \rightarrow e^\Pi$. If we have a connection $g^{-1} dg + \text{Ad}_g^{-1} \theta^c$ in one trivialization, and another $g^{-1} dg + \text{Ad}_g^{-1} \theta$ in the other trivialization, then they will match up only if

$$\theta = \text{Ad}_k \theta^c - dk k^{-1}.$$

If we write $\theta^c = A^c dz + \bar{A}^c d\bar{z}$ and $\theta = A dz + B d\bar{z}$, then we find

$$B = \text{Ad}_k \bar{A}^c - \frac{\partial k}{\partial \bar{z}} k^{-1}.$$

Therefore compatible connections are precisely those with

$$\bar{A}^c = k^{-1} \frac{\partial k}{\partial \bar{z}}, \quad A^c = \bar{k}^{-1} \frac{\partial \bar{k}}{\partial z} + \text{Ad}_k^{-1} A.$$

Solving, we find

$$\begin{aligned} A &= k \operatorname{Ad}_{\bar{k}}^{-1} \frac{\partial \bar{k}}{\partial z} - \frac{\partial k}{\partial z} k^{-1} \\ B &= 0 \\ A^c &= \bar{k}^{-1} \frac{\partial \bar{k}}{\partial z}. \end{aligned}$$

Global existence and uniqueness follows by local existence and uniqueness. \square

Characteristic classes are independent of choice of connection, so we can pick the compatible connection.

Lemma 6. *On E^{LA} , the curvature of the compatible connection has type $(1, 1)$.*

Proof. The $(0, 2)$ part is $\bar{\partial}^2 = 0$, and the $(2, 0)$ part is $\partial^2 = 0$. \square

Therefore the characteristic forms for this connection are (p, p) -forms. On a Kähler manifold, the decomposition of forms descends to a decomposition of cohomology, and a closed (p, p) -form represents a (p, p) -cohomology class. \square

6. THE CHARACTERISTIC RING

We prove theorem 1 on page 2.

Proof. Consider the ring morphism $P(G) \mapsto [P(E)]$. Both rings are obtained as quotients of the ring of invariant polynomials on $\mathfrak{l} \oplus \mathfrak{a}$. So we need to check that every relation satisfied by forms on the model is satisfied in cohomology in every such geometry.

Suppose that $P(G) = 0$ in the model, for some invariant polynomial P . Consider the expression $P(E)$. This agrees entirely with $P(G)$, modulo terms involving the curvature of the Cartan geometry. The Cartan geometry curvature is $(2, 0)$, and semibasic, i.e. multiples of $\omega^{G/P}$. On E^{LA} , $\omega^{G/P}$ remains $(1, 0)$, so these curvature terms remain $(2, 0)$. The 1-form ω^{LA} also remains $(1, 0)$, but the 1-form ω^N can be $(1, 0) + (0, 1)$. Since N is a Lie subalgebra, there are no $\omega^N \wedge \omega^N$ terms in $d\omega^{G/P}$ or $d\omega^{LA}$. So $d\omega^{G/P}$ and $d\omega^{LA}$ are $(2, 0) + (1, 1)$. The Cartan connection curvature terms only appear in the $(2, 0)$ -parts. Therefore the resulting (p, p) -part has no Cartan connection curvature. It must therefore be given by replacing ω^N by its $(0, 1)$ -part in equations for $d\omega^{G/P}$ and $d\omega^{LA}$. However, up on G , the expression for $P(G)$ vanishes with linearly independent 1-forms appearing as ω^N . Therefore, these expressions must vanish on E^{LA} . \square

Lemma 7. *If W is a P -module, we can give rise to an associated vector bundle $E \times_P W \rightarrow M$ on every Cartan geometry modelled on some G/P . If we choose $E = G$, then this gives an isomorphism taking the representation semiring of P to the semiring of G -equivariant vector bundles on G/P . For any choice of Cartan geometry $E \rightarrow M$, we can map G -equivariant vector bundles on G/P to vector bundles on M , via $W \mapsto E \times_P W$. This rule maps characteristic forms to characteristic classes, the same ring morphism as above.*

Proof. The characteristic classes must restrict to each reduction to be given by the restrictions of the characteristic forms. \square

Lemma 8. *Suppose that $E \rightarrow M$ is a Cartan geometry, and M is a closed Kähler manifold. Then the Chern classes of the tangent bundle of M satisfy any relations that are satisfied between the Chern forms of G/P .*

Proof. The tangent bundle is $TM = E \times_P W$ for $W = \mathfrak{g}/\mathfrak{p}$. \square

Example 11 (Affine connections). The characteristic forms of affine space all vanish. Therefore the characteristic classes of any closed Kähler manifold which admits a holomorphic affine connection vanish.

Example 12 (Projective connections). On $G = \mathrm{SL}(n+1, \mathbb{C})$,

$$\nabla^{LA} \omega^{LA} = -(\delta_j^i \omega_k + \delta_k^i \omega_j) \wedge \omega^k.$$

For example, let's look at the projective plane, $n = 2$. Then

$$\nabla^{LA} \omega^{LA} = - \begin{pmatrix} 2\omega_1 \wedge \omega^1 + \omega_2 \wedge \omega^2 & \omega_2 \wedge \omega^1 \\ \omega_1 \wedge \omega^2 & \omega_1 \wedge \omega^1 + 2\omega_2 \wedge \omega^2 \end{pmatrix}.$$

The Chern forms are

$$\begin{aligned} c_1(T) &= \mathrm{tr} \left(\frac{\sqrt{-1}}{2\pi} \nabla^{LA} \omega^{LA} \right) \\ &= -\frac{3\sqrt{-1}}{2\pi} (\omega_1 \wedge \omega^1 + \omega_2 \wedge \omega^2) \\ c_2(T) &= \det \left(\frac{\sqrt{-1}}{2\pi} \nabla^{LA} \omega^{LA} \right) \\ &= -\frac{3}{2\pi^2} \omega_1 \wedge \omega^1 \wedge \omega_2 \wedge \omega^2 \\ &= \frac{1}{3} c_1(T)^2. \end{aligned}$$

This calculation occurs entirely on $G = \mathrm{SL}(3, \mathbb{C})$, but it gives the correct relation among Chern classes for any closed Kähler surface bearing a projective connection.

Theorem 2 (Gunning [15]). *On \mathbb{P}^n , the Chern classes of the tangent bundle T satisfy*

$$c_p(T) = \binom{n+1}{p} \left(\frac{c_1(T)}{n+1} \right)^p.$$

Therefore any closed Kähler manifold M with a holomorphic Cartan geometry modelled on \mathbb{P}^n (a holomorphic projective connection) must satisfy these same equations.

For *normal* Cartan geometries modelled on \mathbb{P}^n , also called normal projective connections, this was proven by Gunning [15]. See Čap [27] for the definition of normalcy. We have generalized Gunning's work, not requiring normalcy, which is vital to his proof, since he works with normal projective connections as objects in a certain sheaf cohomology. Gunning's interpretation of normal projective connections as objects in sheaf cohomology has no known generalization to abnormal projective connections or more general Cartan connections. It would appear difficult and unnatural to attempt to find such a generalization. We are pruning (or weeding) the sheaf cohomology.

Remark 5. One could study secondary characteristic classes of flat Cartan geometries by a similar mechanism.

7. ATIYAH CLASSES

We recall the definition of Atiyah class: any exact sequence of vector bundles

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

induces an exact sequence

$$0 \rightarrow C^* \otimes A \rightarrow C^* \otimes B \rightarrow C^* \otimes C \rightarrow 0.$$

In cohomology, we find the exact sequence

$$\dots \longrightarrow H^0(C^* \otimes C) \xrightarrow{\delta} H^1(C^* \otimes A) \longrightarrow \dots$$

One obvious global section of $C^* \otimes C$ is the identity map 1_C . Fix A and C , and imagine looking for all of the possible choices of vector bundles B to put in the middle. There is an obvious choice: $B = A \oplus C$, which has $\delta 1_C = 0$. It turns out that $\delta 1_C$ determines B up to isomorphism.

For a complex Lie group G and holomorphic principal right G -bundle, $\pi : Z \rightarrow M$, on any complex manifold M , there is a vector bundle on Z of vertical vectors, $\ker \pi'$. At each point $z \in Z$ in the fiber over a point $m \in M$, we find an exact sequence of vector spaces

$$0 \rightarrow \ker \pi'(z) \rightarrow T_Z \rightarrow T_m M \rightarrow 0.$$

We can identify each vertical vector $v \in \ker \pi'(z)$ with a unique vector $A \in \mathfrak{g}$ in the Lie algebra of G : A is the element whose infinitesimal right action on Z is given by the vector v . Hence $\ker \pi'(z) \cong \mathfrak{g}$. This yields an exact sequence of vector bundles on M :

$$0 \rightarrow Z \times_G \mathfrak{g} \rightarrow TZ^G \rightarrow TM \rightarrow 0.$$

where TZ^G is the bundle on M whose local sections are G -invariant vector fields on Z . A holomorphic connection on Z is a choice of splitting of this exact sequence.

The identity element $1_{TM} \in T^*M \otimes TM$ yields an element $a(Z) = \delta 1_{TM}$, called the *Atiyah class* of Z . If there is a holomorphic connection, then clearly $a(Z) = 0$, since the exact sequences split. In fact, Atiyah [1] proves that the exact sequences split, and hence there is a holomorphic connection, just when $a(Z) = 0$.

For a smooth (not necessarily holomorphic) $(1,0)$ -connection θ on Z , define $a(\theta) = (\nabla^\theta \theta)^{(1,1)}$, the $(1,1)$ -part of the curvature. This has a cohomology class $[a(\theta)] \in H^{1,1}(M, Z \times_G \mathfrak{g})$. Atiyah [1] proves that this cohomology class is independent of the choice of $(1,0)$ -connection, and corresponds to $a(Z)$ under the Dolbeault isomorphism

$$H^1(M, \Omega^1(M) \otimes (Z \times_G \mathfrak{g})) \rightarrow H^{1,1}(M, Z \times_G \mathfrak{g}).$$

Let $P \subset G$ be a closed complex subgroup of a complex Lie group. Let $N \subset P$ be a closed normal complex subgroup. For any Cartan geometry $E \rightarrow M$ modelled on G/P , we can construct the Atiyah class $a(E/N)$.

Lemma 9. *Let $E \rightarrow M$ be a Cartan geometry with model G/P , decomposed as $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Suppose that there is a closed normal complex Lie subgroup $N \subset P$, with Lie algebra \mathfrak{n} . Then*

$$a(E/N) = (\nabla^{LA} \omega^{LA})^{(1,1)},$$

for any choice of reduction $E^{LA} \subset E$, under the obvious biholomorphism $E^{LA} \rightarrow E \rightarrow E/N$.

Proof. Atiyah [1] proves that for any smooth connection of type $(1, 0)$ on E/N , the $(1, 1)$ -part of its curvature represents the Atiyah class. \square

In this sense, we can think of $\nabla^{LA}\omega^{LA}$ as a kind of Atiyah class on E , although we have to pullback to a reduction, and then compute the $(1, 1)$ -part, before we can compute with it.

7.1. The associated graded. Define the filtration of any P -representation W by setting $W^{\leq k} = 0$ if $k \leq 0$ and otherwise set $W^{\leq k}$ to be those $w \in W$ for which $nw \in W^{\leq k-1}$. Let $\text{gr } W$ be the associated graded. Hence $\text{gr } W$ is an LA -representation. For $E \rightarrow M$ any Cartan geometry, clearly the characteristic classes of $E \times_P \text{gr } W$ are the same as those of $E \times_P W$. If there is a closed subgroup $N \subset P$, corresponding to the Lie algebra decomposition $P = LAN$, then clearly $E \times_P \text{gr } W \rightarrow M$ is pulled back from $(E/N) \times_{LA} \text{gr } W \rightarrow M$.

8. PARABOLIC GEOMETRIES

8.1. Parabolic subgroups of semisimple Lie groups. If G is a complex semisimple Lie group, and $P \subset G$ is a complex Lie subgroup containing a Borel subgroup, P is called *parabolic*. Wang [28] shows that G/P is closed, and that G contains a maximal compact subgroup $G^c \subset G$ acting transitively on G/P , with $P^c = P \cap G^c$. Moreover, every parabolic subgroup admits a decomposition $P = LAN$ into a product of a semisimple, an abelian, and a unipotent (see Knapp [16] p. 478). A complex Lie subgroup $P \subset G$ of a complex semisimple Lie group is parabolic just when G/P is a rational projective variety.

Because P contains a Borel subgroup B , it is easy to see that the Lie algebras $\mathfrak{b} \subset \mathfrak{p} \subset \mathfrak{g}$ of $B \subset P \subset G$ are each a sum of root spaces of \mathfrak{g} . (For proof of the statements in this paragraph, see Fulton & Harris [13]). We say that a root α *belongs* to \mathfrak{p} if its root space belongs to \mathfrak{p} , and otherwise we say that α is a root *omitted* from \mathfrak{p} . Moreover, the root spaces of roots α of \mathfrak{g} for which both α and $-\alpha$ have root spaces lying in \mathfrak{p} form the Lie algebra \mathfrak{l} of L , while the root spaces of the other roots belonging to \mathfrak{p} form \mathfrak{n} .

Beyond the elementary facts just stated, all we need to know about parabolic subalgebras is in the following lemma.

Lemma 10. *The Killing form inner product $\langle \delta, \beta \rangle$ (where δ is half the sum of omitted roots, and β any root) vanishes just precisely for β a root of the maximal semisimple subalgebra $\mathfrak{l} \subset \mathfrak{p}$.*

Proof. Knapp [16] p. 330, corollary 5.100 gives a completely elementary proof. We give a proof along the same lines, to keep our exposition self-contained. Pick β any positive root. If γ is any omitted root, and $\langle \gamma, \beta \rangle > 0$, then

$$\gamma, \gamma - \beta, \gamma - 2\beta, \dots, \gamma - q\beta = r\beta\gamma$$

is a string of roots ending in the reflection r_β of γ . To start with, γ already contains a positive multiple of an omitted simple negative root. Equivalently, γ has some negative multiple of a positive simple root α_1 , for which $-\alpha_1$ is omitted. Subtracting the positive root β can only make the multiple of α_1 larger negative. Therefore the entire string consists of omitted roots.

If we have an entire β -string of omitted roots, for a positive root β , clearly

$$\begin{aligned}\langle r_\beta \gamma, \beta \rangle &= -\langle r_\beta \gamma, r_\beta \beta \rangle \\ &= -\langle \gamma, \beta \rangle.\end{aligned}$$

Therefore $\langle \gamma, \beta \rangle$ cancels with $\langle r_\beta \gamma, \beta \rangle$ in the sum $\langle \delta, \beta \rangle$. Hence the entire string cancels out of that sum.

It follows that

$$(3) \quad \langle \delta, \beta \rangle = \sum_{\gamma} \langle \gamma, \beta \rangle$$

where the sum is over omitted roots γ for which both $\langle \gamma, \beta \rangle \leq 0$ and for which the other end of the β -string through γ is not omitted. Of course, the terms with $\langle \gamma, \beta \rangle = 0$ cancel out too, so the sum (3) is over omitted roots γ for which both $\langle \gamma, \beta \rangle < 0$ and for which the other end of the β -string through γ is not omitted. In particular, the sum (3) is a sum of negative terms. But there might not be any terms.

If β is a root of \mathfrak{l} , then clearly reflection in β preserves the roots belonging to the parabolic subgroup \mathfrak{p} , and therefore preserves the omitted roots. So the omitted roots will all lie in β -strings, and $\langle \delta, \beta \rangle = 0$ for these roots. On the other hand, if β is not a root of \mathfrak{l} , then β belongs either to the roots of \mathfrak{n} , or lies among the negatives of roots of \mathfrak{n} . We can assume that β is positive, since we only need to show that $\langle \delta, \beta \rangle \neq 0$. Take $\gamma = -\beta$, to see that the sum (3) has at least one negative term. \square

8.2. Characteristic forms and classes on parabolic geometries.

Lemma 11. *Consider a parabolic subgroup $P \subset G$ of a semisimple complex Lie group. A relation between characteristic classes holds on G just when it holds on G^c , and the same relation occurs between characteristic forms on G . If $E \rightarrow M$ is a Cartan geometry modelled on G/P on a closed Kähler manifold M , then $[P(G)] \rightarrow [P(E)]$ is a ring morphism of the characteristic rings in cohomology.*

Proof. Suppose that $[P(G^c)] = 0$ in cohomology, so that $P(G^c) = d\xi$ for some form ξ on G^c . We can average to ensure that ξ is left invariant, and invariant under right $(LA)^c$ -action. Extend ξ from \mathfrak{g}^c to a complex linear form on \mathfrak{g} . By Zariski density, ξ extends to a left invariant holomorphic form on G , invariant under right LA -action, and $P(G^c) = d\xi$ on G .

Define differential forms $P(E)$ and $\xi(E)$ by setting

$$\begin{aligned}P(E) &= P(\nabla^{LA} \omega^{LA}), \\ \xi(E) &= \xi(\omega, \omega, \dots, \omega),\end{aligned}$$

(where the ξ on the right hand side is $\xi \in \Lambda^*(\mathfrak{g}^*)$). Differentiating we see that $P(E) - d\xi(E)$ vanishes modulo the curvature terms. We have seen that the curvature terms in $P(E)$ appear only outside the (p, p) -terms. The same is true for

$$d\xi(E) = \xi(d\omega, \omega, \dots, \omega) - \xi(\omega, d\omega, \dots, \omega) + \dots \pm \xi(\omega, \omega, \dots, d\omega),$$

as the curvature terms arise as $(2, 0)$ -terms inside each $d\omega$ term. Consequently if $[P(G)] = 0$, then $[P(E)] = 0$. \square

Lemma 12. *Let $B \subset G$ be the Borel subgroup of a semisimple Lie group. The rational cohomology of the rational homogeneous variety G/B is generated by Chern*

classes of G -equivariant line bundles $G \times_B W$, for various 1-dimensional B modules W .

Proof. See Bernšteĭn, Gel'fand, & Gel'fand [3, 4]. \square

Lemma 13. *Suppose that $P_- \subset P_+$ are two closed subgroups of a Lie group G , so that we have a fiber bundle map $G/P_- \rightarrow G/P_+$. Let W be a P_+ -module, and $E \rightarrow M$ a Cartan geometry modelled on G/P_+ . Then $E \rightarrow E/P_-$ is a Cartan geometry modelled on G/P_- , called the lift of the Cartan geometry on M . The characteristic forms of $E \times_{P_+} W$ pull back to the characteristic forms of $E \times_{P_-} W$.*

Proof. When we pull back vector bundles, characteristic classes pull back. \square

Lemma 14. *If M is a closed Kähler manifold bearing a parabolic geometry $E \rightarrow M$ modelled on a rational homogeneous variety G/P , then the map $W \mapsto E \times_P W$ on vector bundles yields a unique ring morphism*

$$H^*(G/P, \mathbb{Q}) \rightarrow H^*(M, \mathbb{Q}),$$

sending Chern classes of $G \times_P W \rightarrow E \times_P W$, sending 1 to 1, and sending the Chern classes of the tangent bundle of G/P to the Chern classes of the tangent bundle of M .

Proof. We make lift to assume that $P = B$. We can then restrict attention to sums of line bundles, for which multiplicativity of the total Chern class ensures a ring homomorphism. \square

Remark 6. Note that the Chern classes live in integer cohomology. By mapping the homogeneous vector bundles, not just Chern forms, we can try to map the integer cohomology classes of the model to those of the manifold, assuming the manifold M is closed and Kähler and the model G/P is a rational homogeneous variety. It seems likely that the map on integer cohomology classes gives a ring morphism $H^*(G/P, \mathbb{Z}) \rightarrow H^*(M, \mathbb{Z})$.

Remark 7. All of the nonzero cohomology classes of G/P are (p, p) -classes, so that the Hodge diamond is zero outside the middle column.

8.3. Cohomology rings of some rational homogeneous varieties. Baston and Eastwood [2] explain how to calculate the integer cohomology rings of rational homogeneous varieties using the Weyl group, in terms of Hasse diagrams. The calculations are extraordinarily complicated, roughly due to the enormous size of the Weyl group. We do not need to know the cohomology rings of the various G/P in order to calculate relations among characteristic classes. Therefore (just for illustration) we will only calculate the rational cohomology rings, and only for G/P with G a rank 2 simple group. Keep in mind that these spaces have no torsion in their integer cohomology groups, so that there is little information lost by working with rational coefficients. It won't be necessary for our purposes to find the integer cohomology lattice inside the rational cohomology.

Recall that the Weyl group of G/P means the subset of the Weyl group of G leaving P invariant. This is precisely the Weyl group of the semisimple part of $L \subset P$ (where $P = LAN$), the subgroup generated by reflections in the roots of L .

Lemma 15 (Borel [5]). *The rational cohomology ring of the flag variety G/B is the quotient*

$$H^*(G/B, \mathbb{Q}) = \text{Sym}_{\mathbb{Q}}^*(\mathfrak{h}_{\mathbb{Q}}^*) / \left(\text{Sym}_{\mathbb{Q}}^+(\mathfrak{h}_{\mathbb{Q}}^*)^{W(G)} \right).$$

The numerator is the algebra of rational coefficient polynomials on the rational vector space generated by the coroots. The denominator is the ideal generated by Weyl group invariant polynomials of positive degree. The explicit map taking a polynomial to a cohomology class is as follows: take an element γ of $\mathfrak{h}_{\mathbb{Z}}^*$, i.e. a weight, thought of as a 1-form on the Cartan subgroup H . Any Lie algebra homomorphism $\gamma : \mathfrak{b} \rightarrow \mathbb{C}$ must vanish on the nilpotent part of \mathfrak{b} , so is determined uniquely by $\gamma : \mathfrak{h} \rightarrow \mathbb{C}$. Therefore $\gamma : \mathfrak{h} \rightarrow \mathbb{C}$ determines a unique homomorphism $\gamma : B \rightarrow \mathbb{C}^\times$, and so determines a holomorphic line bundle $G \times_B \mathbb{C}$ on G/B . The local holomorphic sections of this line bundle are functions f taking open sets of G to \mathbb{C} , so that $r_p^* f = \gamma(p)^{-1} f$, where r_p denotes right action by $p \in P$. We can extend γ to a 1-form on \mathfrak{g} by using the Killing form to split apart $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{b}^\perp$, and setting $\gamma = 0$ on \mathfrak{b}^\perp . We then extend γ to a 1-form on G by left invariance, making γ a connection for this line bundle. Taking the first Chern class:

$$c_1(\gamma) = \frac{\sqrt{-1}}{2\pi} d\gamma,$$

we have associated a cohomology class to each weight. This extends uniquely to a ring morphism.

Lemma 16 (Borel [5]). *The cohomology ring of G/P sits inside the cohomology ring of G/B , as the set of elements invariant under the Weyl group of G/P (i.e. under the Weyl group of L , writing $P = LAN$):*

$$H^*(G/P, \mathbb{Q}) = H^*(G/B, \mathbb{Q})^{W(G/P)}.$$

Example 13 (The flag variety $\mathrm{SL}(3, \mathbb{C})/B$). If $G = \mathrm{SL}(3, \mathbb{C})$, then we can label the two simple roots as $\alpha = e_1 - e_2$ and $\beta = e_2 - e_3$, and then the reflections in those roots are

$$\begin{aligned} r_\alpha(\alpha) &= -\alpha, & r_\alpha(\beta) &= \alpha + \beta, \\ r_\beta(\beta) &= -\beta, & r_\beta(\alpha) &= \alpha + \beta. \end{aligned}$$

The cohomology ring is the quotient of the polynomial ring by the ideal generated by the polynomials

$$\begin{aligned} &\alpha^2 + \alpha\beta + \beta^2, \\ &(\alpha - \beta)(2\alpha + \beta)(2\beta + \alpha). \end{aligned}$$

Reduction via a Gröbner basis shows that this cohomology ring is the one given in table 2 on the next page.

The Chern classes of the tangent bundle are computed out by splitting the tangent bundle into a sum of line bundles, one corresponding to each negative root. So

$$\begin{aligned} c(T) &= (1 - \alpha)(1 - \beta)(1 - (\alpha + \beta)) \\ &= 1 - 2(\alpha + \beta) + 2\alpha\beta + \beta^3 \end{aligned}$$

(modulo the polynomials above). Therefore

$$\begin{aligned} c_1 &= -2(\alpha + \beta) \\ c_2 &= 2\alpha\beta \\ c_3 &= \beta^3. \end{aligned}$$

Example 14 (The projective plane). For $G/P = \mathbb{P}^2$, with Dynkin diagram $\times \text{---} \bullet$, (up to reordering of simple roots) β is a root of $L = \text{SL}(2, \mathbb{C})$. Moreover, $\pm\beta$ are the only roots of L . The cohomology ring of \mathbb{P}^2 is given by the subring of $H^*(G/B, \mathbb{Q})$ invariant under reflection in β (see table 2). It turns out that $(2\alpha + \beta)^3$ lies in the ideal. This simple example shows clearly how complicated the calculations get, even for the most elementary semisimple groups and parabolic subgroups. To find the Chern classes of the tangent bundle, we cannot split it into line bundles, but we can lift it to a vector bundle on G/B , and then it splits into line bundles. We leave the calculation to the reader.

Example 15 (The varieties $\text{SO}(5, \mathbb{C})/P = B_2/P = C_2/P = \text{Sp}(4, \mathbb{C})/P$). As we see in table 2, the reflections in the two simple roots of B_2 are

$$\begin{aligned} r_\alpha(\alpha) &= -\alpha, & r_\alpha(\beta) &= 2\alpha + \beta, \\ r_\beta(\beta) &= -\beta, & r_\beta(\alpha) &= \alpha + \beta. \end{aligned}$$

The ideal generated by polynomials invariant under these reflections is generated by the invariant polynomials

$$\begin{aligned} \beta^2 + 2\alpha\beta + 2\alpha^2, \\ \beta^2(2\alpha + \beta)^2, \\ \alpha^2(\alpha + \beta)^2. \end{aligned}$$

The cohomology rings of the various B_2/P are given in table 2.

Table 2: Cohomologies of G/P for G simple of rank 2. The odd degree cohomology groups vanish.

Group	Dynkin diagram	Root lattice	Cohomology	
			Degree	Basis
A_2	$\times \text{---} \times$	$\begin{array}{c} \bullet\beta \\ \times \quad \bullet \\ \times \quad \odot \\ \times \quad \bullet\alpha \end{array}$	2	α, β
			4	$\alpha\beta, \beta^2$
			6	β^3
A_2	$\times \text{---} \bullet$	$\begin{array}{c} \bullet\beta \\ \times \quad \bullet \\ \times \quad \odot \\ \times \quad \bullet\alpha \\ \bullet \end{array}$	2	$2\alpha + \beta$
			4	$(2\alpha + \beta)^2$
$B_2 = C_2$	$\times \rightleftharpoons \times$	$\begin{array}{c} \beta \\ \bullet \quad \bullet \\ \times \quad \odot \quad \bullet\alpha \\ \times \quad \times \quad \times \end{array}$	2	α, β
			4	$\alpha\beta, \beta^2$
			6	$\alpha\beta^2, \beta^3$
			8	$\alpha\beta^3$

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Table: 2 Cohomologies of G/P for G simple of rank 2 (continued)

Group	Dynkin diagram	Root lattice	Cohomology	
$B_2 = C_2$	$\bullet \rightrightarrows \times$	β $\bullet \quad \bullet \quad \bullet$ $\bullet \quad \odot \quad \bullet \quad \alpha$ $\times \quad \times \quad \times$	Degree	Basis
			2	$\alpha + \beta$
			4	α^2
			6	$\alpha^2(\alpha + \beta)$
$B_2 = C_2$	$\times \rightrightarrows \bullet$	β $\bullet \quad \bullet \quad \bullet$ $\times \quad \odot \quad \bullet \quad \alpha$ $\times \quad \times \quad \bullet$	Degree	Basis
			2	$2\alpha + \beta$
			4	β^2
			6	$\beta^2(2\alpha + \beta)$
G_2	$\times \rightleftharpoons \times$	\bullet $\beta \bullet \quad \bullet \quad \bullet \quad \bullet$ $\times \quad \odot \quad \bullet \quad \alpha$ $\times \quad \times \quad \times \quad \times$ \times	Degree	Basis
			2	α, β
			4	$\alpha\beta, \beta^2$
			6	$\alpha\beta^2, \beta^3$
			8	$\alpha\beta^3, \beta^4$
			10	$\alpha\beta^4, \beta^5$
12	$\alpha\beta^5$			
G_2	$\bullet \rightleftharpoons \times$	\bullet $\beta \bullet \quad \bullet \quad \bullet \quad \bullet$ $\bullet \quad \odot \quad \bullet \quad \alpha$ $\times \quad \times \quad \times \quad \times$ \times	Degree	Basis
			2	$3\alpha + 2\beta$
			4	α^2
			6	$\alpha^2(3\alpha + 2\beta)$
			10	$\alpha^4(3\alpha + 2\beta)$
G_2	$\times \rightleftharpoons \bullet$	\bullet $\beta \bullet \quad \bullet \quad \bullet \quad \bullet$ $\times \quad \odot \quad \bullet \quad \alpha$ $\times \quad \times \quad \times \quad \bullet$ \times	Degree	Basis
			2	$2\alpha + \beta$
			4	β^2
			6	$\beta^2(2\alpha + \beta)$
			10	$\beta^4(2\alpha + \beta)$

Example 16 (G_2/P). The roots of G_2 are drawn in table 2. The reflections in the two simple roots are

$$\begin{aligned}
 r_\alpha(\alpha) &= -\alpha, & r_\alpha(\beta) &= 3\alpha + \beta, \\
 r_\beta(\alpha) &= \alpha + \beta, & r_\beta(\beta) &= -\beta.
 \end{aligned}$$

The ideal of Weyl invariant positive degree polynomials is generated by the invariant polynomials:

$$\begin{aligned} &3\alpha^2 + 3\alpha\beta + \beta^2, \\ &\beta^2(3\alpha + 2\beta)^2(\beta + 3\alpha)^2 \end{aligned}$$

8.4. Relations on characteristic classes of rational homogeneous varieties.

Lemma 17. *Let G be a complex semisimple Lie group with Borel subgroup $B \subset G$. If W is a B -module, then the characteristic forms of the vector bundle $G \times_B W$ are identical to those of $G \times_B \text{gr } W$. Moreover, $G \times_B \text{gr } W$ is a sum of homogeneous line bundles, given by the direct sum*

$$\text{gr } W = \bigoplus_{\lambda} W_{\lambda},$$

with λ a weight, and W_{λ} the associated weight space. The total Chern class of W is therefore

$$c(W) = \prod_{\lambda} \left(1 + \frac{\sqrt{-1}}{2\pi} d\lambda \right)^{\dim W_{\lambda}},$$

where we treat each weight $\lambda \in \mathfrak{h}^* \subset \mathfrak{g}^*$ as a 1-form on G before taking exterior derivative.

8.5. Applications to parabolic geometries. Table 3 on page 24 contains all rational homogeneous varieties G/P of dimension at most 7 with G simple. I have contacted several experts on the cohomology of rational homogeneous varieties, who inform me that the relations among Chern classes of the tangent bundles of rational homogeneous varieties are not known. Being unable to find a simple expression for these relations, I offer the reader this long table to provide the only known relations. Beside each Dynkin diagram are the ranks of the filtration of $\mathfrak{g}/\mathfrak{p}$ so that the tangent bundle of the rational homogeneous variety is invariantly filtered by homogeneous vector subbundles of those ranks. The final column presents lists of polynomials in Chern classes. These polynomials vanish. The notation is somewhat confusing, but has certain advantages. The quantities c_1, c_2, \dots, c_n are the Chern classes of the tangent bundle of G/P . For each G/P , we have found a weight ε of \mathfrak{g} , invariant under the Weyl group of the maximal semisimple subgroup $L \subset P$. Therefore there is a line bundle on G/P associated to this weight. We have expressed c_1 as a multiple of ε , i.e. as a multiple of the first Chern class of the associated line bundle. However, if $c_1 = \varepsilon$, then we have omitted this equation. The relations between Chern classes are far simpler when expressed in terms of this ε .

The relations on Chern classes listed might not generate all such relations. The pattern in these relations is still unclear. The polynomial expressions vanish in cohomology, but many of them do not vanish as invariant differential forms, i.e. the Chern *forms* often do *not* satisfy these polynomials; only the Chern *classes* do. All of the polynomials are exterior derivatives of invariant differential forms. For many G/P , all Chern classes can be expressed in terms of ε (and therefore in terms of c_1). However, in the case of $\times \rightrightarrows \times$, c_3 is *not* a polynomial function of c_1 (or of c_1 and c_2) in the cohomology ring. So in general, we cannot always express Chern classes of rational homogeneous varieties in terms of the first Chern class.

Kobayashi [17] proved that n -dimensional closed Kähler manifolds with conformal geometries satisfy

$$c_p = a_p \left(\frac{c_1}{n} \right)^p, \quad p = 2, 3, \dots, n,$$

for some integers a_p which remain unknown. We have seen above similar relations for projective connections. These are the only relations known outside of our table. It seems likely that a complete description of the relations among Chern classes of rational homogeneous varieties will soon be discovered. Until these relations are known, applications of these results will be restricted to low dimensions. There is as yet no way to see if a compact Kähler manifold has a holomorphic parabolic geometry, other than to write one down. It is possible that the relations on characteristic classes are necessary and sufficient conditions.

Table 3: Chern class relations of some rational homogeneous varieties. These are the varieties G/P of dimension up to 7 with G simple. The dimension of the variety is the last integer in the grading. Notation for rational homogeneous varieties: $\mathbb{F}_{p,q,\dots}^r$ is the space of partial flags of dimensions p, q, \dots in \mathbb{C}^r , $\text{Lag}_{p,q,\dots}^r$ the analogous space of partial subLagrangian flags, $\text{Null}_{p,q,\dots}^r$ the analogous space of partial null flags.

Dynkin Diagram	Model	Grading	Chern Class Relations
Projective connection on a curve \times	\mathbb{P}^1	1	$c_1 - 2\varepsilon$
Projective connection on a surface $\times \text{---} \bullet$	\mathbb{P}^2	2	$c_1 - 3\varepsilon$ $c_2 - 3\varepsilon^2$
Scalar 2nd order ODE $\times \text{---} \times$	$\mathbb{P}T\mathbb{P}^2$	2,3	$c_1 - 2\varepsilon$ $c_2 - 2\varepsilon^2$ $c_3 - \varepsilon^3$
Contact path geometry on 3-fold $\bullet \rightrightarrows \times$	$\text{Lag}_1 \mathbb{C}^4$	2,3	$c_1 - 4\varepsilon$ $c_2 - 6\varepsilon^2$ $c_3 - 4\varepsilon^3$
Conformal geometry on 3-fold $\times \rightrightarrows \bullet$	Q^3	3	$c_1 - 3\varepsilon$ $c_2 - 4\varepsilon^2$ $c_3 - 2\varepsilon^3$
Projective connection on 3-fold $\times \text{---} \bullet \text{---} \bullet$	\mathbb{P}^3	3	$c_1 - 4\varepsilon$ $c_2 - 6\varepsilon^2$ $c_3 - 4\varepsilon^3$
Scalar 3rd order ODE $\times \rightrightarrows \times$	$\text{Lag}_{1,2} \mathbb{C}^4$	2,3,4	$c_1 - 2\varepsilon$ $c_2 - 2\varepsilon^2$ $6c_3\varepsilon - 7\varepsilon^4$ $3c_4 - \varepsilon^4$
Grassmann geometry on 4-fold $\bullet \text{---} \times \text{---} \bullet$	$\text{Gr}_2 \mathbb{C}^4$	4	$c_1 - 4\varepsilon$ $c_2 - 7\varepsilon^2$ $c_3 - 6\varepsilon^3$ $c_4 - 3\varepsilon^4$
Projective connection on 4-fold $\times \text{---} \bullet \text{---} \bullet \text{---} \bullet$	\mathbb{P}^4	4	$c_1 - 5\varepsilon$ $c_2 - 10\varepsilon^2$ $c_3 - 10\varepsilon^3$ $c_4 - 5\varepsilon^4$
$\bullet \leftrightsquigarrow \times$	Q^5	4,5	$c_1 - 3\varepsilon$ $3c_2 - 13\varepsilon^2$ $3c_3 - 11\varepsilon^3$ $3c_4 - 5\varepsilon^4$ $3c_5 - \varepsilon^5$
Nondegenerate 2-plane field on 5-fold $\times \leftrightsquigarrow \bullet$		2,3,5	$c_1 - 5\varepsilon$ $c_2 - 11\varepsilon^2$ $c_3 - 13\varepsilon^3$ $c_4 - 9\varepsilon^4$ $c_5 - 3\varepsilon^5$
$\times \text{---} \bullet \text{---} \times$	$\mathbb{F}_{1,3} \mathbb{C}^4$	4,5	$c_1 - 3\varepsilon$ $5c_2\varepsilon^2 - 21\varepsilon^4$ $c_3 - 2c_2\varepsilon + 5\varepsilon^3$ $5c_4 - 9\varepsilon^4$ $5c_5 - 3\varepsilon^5$ $5c_2^2 - 89\varepsilon^4$

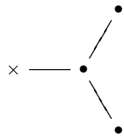

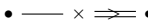




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Table 3: Chern class relations of some rational homogeneous varieties (continued)

Dynkin Diagram	Model	Grading	Chern Class Relations
2nd order ODE, 2 independent variables $\times \text{---} \times \text{---} \bullet$	$\mathbb{P}T\mathbb{P}^3$	3,5	$375 c_1^3 c_2 - 179 c_1^5$ $-51 c_1^2 c_2 + 17 c_1^4 + 54 c_1 c_3$ $18 c_4 - 3 c_1^2 c_2 + c_1^4$ $375 c_5 - c_1^5$ $81 c_2^2 - 66 c_1^2 c_2 + 13 c_1^4$ $1125 c_2 c_3 - 73 c_1^5$
Conformal geometry on 5-fold $\times \text{---} \bullet \Rightarrow \bullet$	Q^5	5	$c_1 - 5 \varepsilon$ $c_2 - 11 \varepsilon^2$ $c_3 - 13 \varepsilon^3$ $c_4 - 9 \varepsilon^4$ $c_5 - 3 \varepsilon^5$
Contact path geometry on 5-fold $\times \text{---} \bullet \Leftarrow \bullet$	$\text{Lag}_1 \mathbb{C}^6$	4,5	$c_1 - 6 \varepsilon$ $c_2 - 15 \varepsilon^2$ $c_3 - 20 \varepsilon^3$ $c_4 - 15 \varepsilon^4$ $c_5 - 6 \varepsilon^5$
Projective connection on 5-fold $\times \text{---} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet$	\mathbb{P}^5	5	$c_1 - 6 \varepsilon$ $c_2 - 15 \varepsilon^2$ $c_3 - 20 \varepsilon^3$ $c_4 - 15 \varepsilon^4$ $c_5 - 6 \varepsilon^5$
$\times \Leftarrow \Leftarrow \times$		2,3,4,5,6	$c_1 - 2 \varepsilon$ $c_2 - 2 \varepsilon^2$ $180 c_3 \varepsilon^3 - 227 \varepsilon^6$ $c_4 - 2 c_3 \varepsilon + 2 \varepsilon^4$ $49 c_5 - 48 c_3 \varepsilon^2 + 54 \varepsilon^5$ $60 c_6 - \varepsilon^6$ $90 c_3^2 - 143 \varepsilon^6$
$\times \text{---} \times \text{---} \times$	$\mathbb{F}_{1,2,3} \mathbb{C}^4$	3,5,6	$c_1 - 2 \varepsilon$ $c_2 - 2 \varepsilon^2$ $18 c_3 \varepsilon^3 - 23 \varepsilon^6$ $c_4 - 2 c_3 \varepsilon + 2 \varepsilon^4$ $5 c_5 - 3 c_3 \varepsilon^2 + 3 \varepsilon^5$ $30 c_6 - \varepsilon^6$ $45 c_3^2 - 73 \varepsilon^6$
Nondegenerate 3-plane field on 6-fold $\bullet \text{---} \bullet \Rightarrow \times$	$\text{Null}_3 \mathbb{C}^7$	3,6	$c_1 - 6 \varepsilon$ $c_2 - 16 \varepsilon^2$ $c_3 - 24 \varepsilon^3$ $c_4 - 22 \varepsilon^4$ $c_5 - 12 \varepsilon^5$ $c_6 - 4 \varepsilon^6$
$\bullet \text{---} \bullet \Leftarrow \times$	$\text{Lag}_3 \mathbb{C}^6$	6	$c_1 - 4 \varepsilon$ $2 c_2 - 15 \varepsilon^2$ $8 c_3 \varepsilon - 67 \varepsilon^4$ $4 c_4 - 23 \varepsilon^4$ $4 c_5 - 9 \varepsilon^5$ $2 c_6 - \varepsilon^6$ $8 c_3^2 - 555 \varepsilon^6$
Grassmann geometry on 6-fold $\bullet \text{---} \times \text{---} \bullet \text{---} \bullet$	$\text{Gr}_2 \mathbb{C}^5$	6	$c_1 - 5 \varepsilon$ $5 \varepsilon^3 c_2 - 57 \varepsilon^5$ $c_3 - 15 \varepsilon^3$ $c_4 - 5 c_2 \varepsilon^2 + 45 \varepsilon^4$ $c_5 - 6 \varepsilon^5$ $c_6 - 2 \varepsilon^6$ $c_2^2 - 25 c_2 \varepsilon^2 + 155 \varepsilon^4$

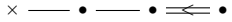

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Table 3: Chern class relations of some rational homogeneous varieties (continued)

Dynkin Diagram	Model	Grading	Chern Class Relations
<p>Conformal geometry on 6-fold</p> 	Q^6	6	$\begin{aligned} c_1 - 6 \varepsilon \\ c_2 - 16 \varepsilon^2 \\ c_3 - 24 \varepsilon^3 \\ c_4 - 22 \varepsilon^4 \\ c_5 - 12 \varepsilon^5 \\ c_6 - 4 \varepsilon^6 \end{aligned}$
<p>Projective connection on 6-fold</p> 	\mathbb{P}^6	6	$\begin{aligned} c_1 - 7 \varepsilon \\ c_2 - 21 \varepsilon^2 \\ c_3 - 35 \varepsilon^3 \\ c_4 - 35 \varepsilon^4 \\ c_5 - 21 \varepsilon^5 \\ c_6 - 7 \varepsilon^6 \end{aligned}$
	$\text{Null}_2 \mathbb{C}^7$	6,7	$\begin{aligned} c_1 - 4 \varepsilon \\ 14 \varepsilon^4 c_2 - 107 \varepsilon^6 \\ c_3 - 3 c_2 \varepsilon + 14 \varepsilon^3 \\ 3 c_4 + c_2 \varepsilon^2 - 28 \varepsilon^4 \\ 3 c_5 + 17 \varepsilon^3 c_2 - 140 \varepsilon^5 \\ 14 c_6 - 15 \varepsilon^6 \\ 14 c_7 - 3 \varepsilon^7 \\ 3 c_2^2 - 44 c_2 \varepsilon^2 + 161 \varepsilon^4 \end{aligned}$
<p>Nondegenerate 4-plane field on 7-fold, a.k.a. quaternionic contact structure</p> 	$\text{Lag}_2 \mathbb{C}^6$	4,7	$\begin{aligned} c_1 - 5 \varepsilon \\ 7 \varepsilon^4 c_2 - 82 \varepsilon^6 \\ c_3 - 2 c_2 \varepsilon + 7 \varepsilon^3 \\ 3 c_4 - 5 c_2 \varepsilon^2 + 14 \varepsilon^4 \\ 3 c_5 - 10 \varepsilon^3 c_2 + 91 \varepsilon^5 \\ 7 c_6 - 24 \varepsilon^6 \\ 7 c_7 - 6 \varepsilon^7 \\ 3 c_2^2 - 74 c_2 \varepsilon^2 + 455 \varepsilon^4 \end{aligned}$
	$\mathbb{F}_{1,4} \mathbb{C}^5$	6,7	$\begin{aligned} c_1 - 4 \varepsilon \\ 7 \varepsilon^4 c_2 - 52 \varepsilon^6 \\ c_3 - 3 c_2 \varepsilon + 14 \varepsilon^3 \\ c_4 - 5 c_2 \varepsilon^2 + 31 \varepsilon^4 \\ c_5 - 5 \varepsilon^3 c_2 + 34 \varepsilon^5 \\ 7 c_6 - 8 \varepsilon^6 \\ 7 c_7 - 2 \varepsilon^7 \\ c_2^2 - 17 c_2 \varepsilon^2 + 71 \varepsilon^4 \end{aligned}$
	$\mathbb{F}_{1,2} \mathbb{C}^5$	4,7	$\begin{aligned} c_1 - 2 \varepsilon \\ 126 c_2 \varepsilon^5 - 239 \varepsilon^7 \\ -3942 \varepsilon^4 c_2 + 2160 c_3 \varepsilon^3 + 5083 \varepsilon^6 \\ 768 c_4 \varepsilon - 816 c_3 \varepsilon^2 + 414 \varepsilon^3 c_2 - 215 \varepsilon^5 \\ 192 c_5 - 48 c_3 \varepsilon^2 - 18 \varepsilon^3 c_2 + 65 \varepsilon^5 \\ 540 c_6 - 54 \varepsilon^4 c_2 + 91 \varepsilon^6 \\ 378 c_7 - \varepsilon^7 \\ c_2^2 - c_4 + 2 c_3 \varepsilon - 6 c_2 \varepsilon^2 + 6 \varepsilon^4 \\ 1536 c_2 c_3 - 1104 c_3 \varepsilon^2 - 4878 \varepsilon^3 c_2 + 7247 \varepsilon^5 \\ 4320 c_2 c_4 - 12582 \varepsilon^4 c_2 + 20303 \varepsilon^6 \\ 4320 c_3^2 - 16794 \varepsilon^4 c_2 + 26561 \varepsilon^6 \\ 378 c_3 c_4 - 181 \varepsilon^7 \end{aligned}$
<p>Conformal geometry on 7-fold</p> 	Q^7	7	$\begin{aligned} c_1 - 7 \varepsilon \\ c_2 - 22 \varepsilon^2 \\ c_3 - 40 \varepsilon^3 \\ c_4 - 46 \varepsilon^4 \\ c_5 - 34 \varepsilon^5 \\ c_6 - 16 \varepsilon^6 \\ c_7 - 4 \varepsilon^7 \end{aligned}$

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Table 3: Chern class relations of some rational homogeneous varieties (continued)

Dynkin Diagram	Model	Grading	Chern Class Relations
<p>Contact path geometry on 7-fold</p> 	$\text{Lag}_1 \mathbb{C}^8$	6,7	$c_1 - 8 \varepsilon$ $c_2 - 28 \varepsilon^2$ $c_3 - 56 \varepsilon^3$ $c_4 - 70 \varepsilon^4$ $c_5 - 56 \varepsilon^5$ $c_6 - 28 \varepsilon^6$ $c_7 - 8 \varepsilon^7$
<p>Projective connection on 7-fold</p> 	\mathbb{P}^7	7	$c_1 - 8 \varepsilon$ $c_2 - 28 \varepsilon^2$ $c_3 - 56 \varepsilon^3$ $c_4 - 70 \varepsilon^4$ $c_5 - 56 \varepsilon^5$ $c_6 - 28 \varepsilon^6$ $c_7 - 8 \varepsilon^7$

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