

# HOLOMORPHIC CARTAN GEOMETRIES AND CALABI–YAU MANIFOLDS

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ABSTRACT. We prove that the only Calabi–Yau projective manifolds which bear holomorphic Cartan geometries are precisely the abelian varieties.

RÉSUMÉ. Nous démontrons que les seules variétés projectives de Calabi–Yau qui possèdent des géométries holomorphes de Cartan sont les variétés abéliennes.

## 1. INTRODUCTION

Let  $G$  be a complex Lie group and  $H \subset G$  a closed complex subgroup. The Lie algebras of  $G$  and  $H$  will be denoted by  $\mathfrak{g}$  and  $\mathfrak{h}$  respectively.

Let  $M$  be a connected complex manifold. Let

$$E_H \longrightarrow M$$

be a holomorphic principal right  $H$ –bundle. For each point  $h \in H$ , let

$$(1.1) \quad \tau_h : E_H \longrightarrow E_H$$

be the translation by the action of  $h$ . For any  $v \in \mathfrak{h}$ , let  $\zeta_v$  be the holomorphic vector field on  $E_H$  that assigns to  $z \in E_H$  the tangent vector defined by the analytic curve

$$t \longmapsto z \exp(tv).$$

A *holomorphic Cartan geometry* on  $M$  of type  $G/H$  is a holomorphic principal  $H$ –bundle

$$(1.2) \quad E_H \longrightarrow M$$

together with a holomorphic 1–form on  $E_H$  with values in  $\mathfrak{g}$

$$(1.3) \quad \omega \in H^0(E_H, \Omega_{E_H}^1 \otimes_{\mathbb{C}} \mathfrak{g})$$

satisfying the following three conditions:

- $\omega(z)(\zeta_v(z)) = v$  for all  $v \in \mathfrak{h}$  and  $z \in E_H$ , where  $\zeta_v$  is the vector field defined above,
- $\tau_h^* \omega = \text{Ad}(h^{-1}) \circ \omega$  for all  $h \in H$ , where  $\tau_h$  is defined in (1.1), and
- for each point  $z \in E_H$ , the homomorphism from the holomorphic tangent space

$$(1.4) \quad \omega(z) : T_z E_H \longrightarrow \mathfrak{g}$$

is an isomorphism of vector spaces.

All Chern classes will have coefficients in  $\mathbb{Q}$ .

By a Calabi–Yau manifold we mean a compact Kähler manifold  $M$  with  $c_1(M) = 0$ .

Sorin Dumitrescu [5] proved that if a Calabi–Yau manifold has a rigid holomorphic geometric structure of affine algebraic type, then it admits a finite covering by a torus. In [6] the second author conjectured a more general result that if a Calabi–Yau manifold  $M$  admits a holomorphic Cartan geometry, then  $M$  is covered by a torus. Our aim here is to prove this conjecture under the assumption that  $M$  is complex projective.

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## 2. FLAT VECTOR BUNDLES ON A CALABI–YAU MANIFOLD

Let  $M$  be a Calabi–Yau manifold. We will later impose the condition that  $M$  is projective.

**Lemma 2.1.** *Let  $(E, \nabla)$  be a flat vector bundle over  $M$ . Let*

$$E \longrightarrow Q \longrightarrow 0$$

*be a quotient holomorphic vector bundle such that  $c_1(Q) = 0$ . Then*

$$c_i(Q) = 0$$

*for all  $i \geq 1$ .*

*Proof.* It is well known [2, p. 764, Théorème 2(2)] that there is a finite étale Galois covering

$$(2.1) \quad \widetilde{M} \longrightarrow M$$

by a product  $\widetilde{M} = M_{CY} \times T^n$  of a simply connected Calabi–Yau manifold with a complex torus. (Note that  $X_j$  in [2, Théorème 2(2)] are simply connected [2, p. 763, Proposition 4(2)].) Clearly we can replace  $M$  by  $\widetilde{M}$  without loss of generality, and so assume that the fundamental group  $\pi_1(M)$  is  $\mathbb{Z}^n$ . The flat vector bundle  $E$  is given by a linear representation of  $\mathbb{Z}^n$ . Since  $\mathbb{Z}^n$  is abelian, any complex linear representation of  $\mathbb{Z}^n$  admits a filtration such that each successive quotient is a complex one-dimensional  $\mathbb{Z}^n$ -module. Therefore, we have a filtration of  $E$  by holomorphic subbundles

$$(2.2) \quad 0 = E_0 \subset E_1 \subset \cdots \subset E_{\ell-1} \subset E_\ell = E$$

such that each successive quotient  $E_i/E_{i-1}$ ,  $i \in [1, \ell]$ , is a flat line bundle.

Therefore,  $E$  is numerically flat [4, p. 311, Theorem 1.18]. In particular,  $E$  is numerically effective. Hence  $Q$  is numerically effective [4, P. 308, Proposition 1.15(i)]. Since  $c_1(Q) = 0$ , this implies that  $E$  is numerically flat (see [4, p. 311, Definition 1.17]). Consequently,  $c_i(Q) = 0$  for all  $i \geq 1$  [4, p. 311, Corollary 1.19].  $\square$

Fix any Kähler class on  $M$ . By Yau’s proof of Calabi’s conjecture [7],  $M$  admits a Ricci flat Kähler metric, say  $\theta$ , in the given Kähler class. Note that the condition that  $\theta$  is Ricci flat implies that the holomorphic tangent bundle  $TM$  is polystable of degree zero with respect to  $\theta$  (i.e., a direct sum of stable vector bundles of degree zero).

**Lemma 2.2.** *Let  $M$  be a Calabi–Yau complex projective manifold. Let  $E \rightarrow M$  be a holomorphic vector bundle with a holomorphic connection. Then  $E$  admits a holomorphic flat connection.*

*Proof.* This was proved in [3] (see [3, p. 2827, Theorem A(1)]). Note that  $TM$  is semistable because it is polystable. Therefore, [3, Theorem A(1)] applies.  $\square$

### 3. CARTAN GEOMETRIES AND CALABI–YAU MANIFOLDS

As before,  $M$  is a Calabi–Yau projective manifold. Let  $G$  be a complex Lie group, and let  $H \subset G$  be a closed complex subgroup. Let  $(E_H, \omega)$  be a pair as in (1.2) and (1.3) defining a Cartan geometry of type  $G/H$  on  $M$ .

Let

$$(3.1) \quad E_G := E_H \times_H G \rightarrow M$$

be the holomorphic principal  $G$ -bundle obtained by extending the structure group of the principal  $H$ -bundle  $E_H$  using the inclusion map of  $H$  in  $G$ . The form  $\omega$  defines a holomorphic connection on  $E_G$ . We recall the construction of this holomorphic connection.

Let  $\omega_{MC} : TG \rightarrow G \times \mathfrak{g}$  be the  $\mathfrak{g}$ -valued Maurer–Cartan one-form on  $G$  constructed using the left invariant vector fields. Consider the  $\mathfrak{g}$ -valued holomorphic one-form

$$\tilde{\omega} := p_1^* \omega + p_2^* \omega_{MC}$$

on  $E_H \times G$ , where  $p_1$  (respectively,  $p_2$ ) is the projection of  $E_H \times G$  to  $E_H$  (respectively,  $G$ ). This form  $\tilde{\omega}$  descends to a  $\mathfrak{g}$ -valued holomorphic 1-form on the quotient space  $E_G$  in (3.1). This descended 1-form defines a holomorphic connection on  $E_G$ .

**Theorem 3.1.** *Any Calabi–Yau complex projective manifold  $M$  bearing a holomorphic Cartan geometry is holomorphically covered by an abelian variety.*

*Proof.* Let  $\text{ad}(E_H)$  and  $\text{ad}(E_G)$  be the adjoint vector bundles of  $E_H$  and  $E_G$  respectively. Since the homomorphism in (1.4) is an isomorphism, it follows that

$$(3.2) \quad TM = \text{ad}(E_G)/\text{ad}(E_H).$$

We noted above that  $E_G$  is equipped with a holomorphic connection. A holomorphic connection on  $E_G$  defines a holomorphic connection on  $\text{ad}(E_G)$ . Hence  $\text{ad}(E_G)$  admits a holomorphic connection. So from Lemma 2.2 it follows that  $\text{ad}(E_G)$  admits a flat holomorphic connection. In view of Lemma 2.1, from (3.2) we conclude that  $c_2(TM) = 0$ . Now the proof is completed by Lemma 1 of [6].  $\square$

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