

HOLOMORPHIC CARTAN GEOMETRIES, CALABI–YAU MANIFOLDS AND RATIONAL CURVES

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ABSTRACT. We prove that if a Calabi–Yau manifold M admits a holomorphic Cartan geometry, then M is covered by a complex torus. This is done by establishing the Bogomolov inequality for semistable sheaves on compact Kähler manifolds. We also classify all holomorphic Cartan geometries on rationally connected projective varieties.

1. INTRODUCTION

In [14], the second author conjectured that the only Calabi–Yau manifolds that bear holomorphic Cartan geometries are those covered by the complex tori. Our first theorem confirms this conjecture. More precisely, we prove:

Theorem 1.1. *Let M be a compact connected Kähler manifold with $c_1(M) = 0$. If M is equipped with a holomorphic Cartan geometry, then there is an étale Galois holomorphic covering map*

$$A \longrightarrow M,$$

where A is a complex torus.

We then consider Cartan geometries on rationally connected projective varieties.

Let G be a complex Lie group and $H \subset G$ a closed complex subgroup. The following

Theorem 1.2. *Let M be a rationally connected complex projective manifold and (E_H, θ) a holomorphic Cartan geometry of type G/H on M . Then G/H is a rational homogeneous space, and $M \cong G/H$. Furthermore, (M, E_H, θ) is holomorphically isomorphic to the tautological parabolic geometry $(G/H, G, \theta_0)$.*

While we were finalizing this work, a preprint [8] appeared where Theorem 1.1 is proved by different methods. Theorem 1.2 is known for parabolic geometries [13], [5].

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2. BOGOMOLOV INEQUALITY

Let M be a compact connected Kähler manifold equipped with a Kähler form ω .

For a torsionfree coherent analytic sheaf $V \neq 0$ on M , define

$$\text{degree}(V) := (c_1(V) \wedge \omega^{d-1}) \cap [M] \in \mathbb{R},$$

where $d = \dim_{\mathbb{C}} M$. We recall that V is called *semistable* (respectively, *stable*) if for all torsionfree coherent analytic sheaves $V' \subset V$ with $\text{rank}(V') < \text{rank}(V)$, the inequality

$$\frac{\text{degree}(V')}{\text{rank}(V')} \leq \frac{\text{degree}(V)}{\text{rank}(V)} \quad (\text{respectively, } \frac{\text{degree}(V')}{\text{rank}(V')} < \frac{\text{degree}(V)}{\text{rank}(V)})$$

holds. A semistable sheaf V is called *polystable* if it is a direct sum of stable sheaves.

Lemma 2.1. *Let V be a semistable sheaf on M . Then*

$$((2r \cdot c_2(V) - (r-1)c_1(V)^2) \cup \omega^{d-2}) \cap [M] \geq 0,$$

where r is the rank of V .

Proof. If V is polystable and reflexive, then this is proved in [2] (see [2, p. 40, Corollary 3]).

Assume that V is polystable.

The double dual V^{**} is reflexive ([10, p. 160, Proposition (5.18)]). Since V is torsionfree, the natural homomorphism $V \rightarrow V^{**}$ is injective. Hence we have a short exact sequence of coherent analytic sheaves

$$(2.1) \quad 0 \rightarrow V \rightarrow V^{**} \rightarrow T \rightarrow 0$$

on M , where T is a torsion sheaf, and the codimension of the support of T is at least two (see [10, p. 154, Proposition (4.14)]).

Since V is polystable, we know that V^{**} is also polystable (follows from the fact that the double dual of a stable sheaf is stable). Hence

$$(2.2) \quad ((2r \cdot c_2(V^{**}) - (r-1)c_1(V^{**})^2) \cup \omega^{d-2}) \cap [M] \geq 0$$

[2, p. 40, Corollary 3].

Consider the quotient T in (2.1). Let S_1, \dots, S_ℓ be the components of the support of T of codimension two. Let r_i be the rank of $T|_{S_i}$. Using (2.1) we conclude that

$$(2.3) \quad c_1(V) = c_1(V^{**}) \quad \text{and} \quad c_2(V) = c_2(V^{**}) + \sum_{i=1}^{\ell} r_i \cdot \psi([S_i]),$$

where $\psi : H_{2d-4}(M, \mathbb{Q}) \rightarrow H^4(M, \mathbb{Q})$ is the isomorphism given by the Poincaré duality.

Since $\text{Volume}_{\omega}(S_i) = (\omega^{d-2} \cup \psi([S_i])) \cap [M] > 0$, from (2.2) and (2.3) we conclude that

$$(2.4) \quad ((2r \cdot c_2(V) - (r-1)c_1(V)^2) \cup \omega^{d-2}) \cap [M] \geq 0.$$

Now let V be any semistable sheaf. From [10, p. 175, Theorem (7.18)], we know that V has a filtration of coherent analytic subsheaves

$$0 =: V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$$

such that each successive quotient V_i/V_{i-1} , $i \in [1, n]$, is torsionfree and stable, and furthermore,

$$\frac{\text{degree}(V_i/V_{i-1})}{\text{rank}(V_i/V_{i-1})} = \frac{\text{degree}(V)}{\text{rank}(V)}.$$

Therefore,

$$\widehat{V} := \bigoplus_{i=1}^n V_i/V_{i-1}$$

is a polystable sheaf. Since

$$c_i(V) = c_i(\widehat{V})$$

for all $i \geq 0$, from (2.4) we conclude that the inequality in the lemma holds. This completes the proof of the lemma. \square

3. CARTAN GEOMETRIES AND CALABI–YAU MANIFOLDS

Let G be a complex Lie group and $H \subset G$ a closed complex subgroup. As before, M is a compact connected Kähler manifold.

A *holomorphic Cartan geometry* on M of type G/H is a holomorphic principal H -bundle

$$(3.1) \quad E_H \longrightarrow M$$

together with a holomorphic one-form on E_H with values in \mathfrak{g}

$$(3.2) \quad \theta \in H^0(E_H, \Omega_{E_H}^1 \otimes_{\mathbb{C}} \mathfrak{g})$$

satisfying the following three conditions:

- $\theta(z)(\zeta_v(z)) = v$ for all $v \in \mathfrak{h}$ and $z \in E_H$, where ζ_v is the vector field defined above,
- $\tau_h^* \theta = \text{Ad}(h^{-1}) \circ \theta$ for all $h \in H$, where τ_h is defined in (2.1), and
- for each point $z \in E_H$, the homomorphism from the holomorphic tangent space

$$(3.3) \quad \theta(z) : T_z E_H \longrightarrow \mathfrak{g}$$

is an isomorphism of vector spaces.

(See [15].)

Let (E_H, θ) be a holomorphic Cartan geometry of type G/H on M . Let

$$(3.4) \quad E_G := (E_H \times G)/H \longrightarrow M$$

be the holomorphic principal G -bundle obtained by extending the structure group of E_H using the inclusion of H in G (the action of $h \in H$ sends any $(z, g) \in E_H \times G$ to $(zh, h^{-1}g)$). The Lie algebra of G will be denoted by \mathfrak{g} . Let

$$\theta_{\text{MC}} : TG \longrightarrow G \times \mathfrak{g}$$

be the \mathfrak{g} -valued Maurer–Cartan one-form on G constructed using the left invariant vector fields. Consider the \mathfrak{g} -valued holomorphic one-form

$$\tilde{\omega} := p_1^*\theta + p_2^*\theta_{\text{MC}}$$

on $E_H \times G$, where p_1 (respectively, p_2) is the projection of $E_H \times G$ to E_H (respectively, G). This form $\tilde{\omega}$ descends to a \mathfrak{g} -valued holomorphic one-form on the quotient space E_G in (3.4). This descended one-form defines a holomorphic connection on E_G .

The Lie algebra of H will be denoted by \mathfrak{h} . Consider the principal G -bundle E_G in (3.4). Let $\text{ad}(E_G)$ (respectively, $\text{ad}(E_H)$) be the adjoint bundle of E_G (respectively, E_H). So $\text{ad}(E_G)$ (respectively, $\text{ad}(E_H)$) is associated to E_G for the adjoint action of G (respectively, H) on \mathfrak{g} (respectively, \mathfrak{h}). There is a natural inclusion

$$\text{ad}(E_H) \hookrightarrow \text{ad}(E_G).$$

Using the form θ , the quotient $\text{ad}(E_G)/\text{ad}(E_H)$ gets identified with the holomorphic tangent bundle TM . Therefore, we get a short exact sequence of vector bundle on M

$$(3.5) \quad 0 \longrightarrow \text{ad}(E_H) \longrightarrow \text{ad}(E_G) \longrightarrow TM \longrightarrow 0.$$

Theorem 3.1. *Assume that*

- $c_1(M) = 0$, and
- M is equipped with a holomorphic Cartan geometry (E_H, θ) of type G/H .

Then there is an étale Galois holomorphic covering map

$$\gamma : A \longrightarrow M,$$

where A is a complex torus.

Proof. Consider the principal G -bundle E_G in (3.4). Recall that θ defines a holomorphic connection on E_G . A holomorphic connection on E_G induces a holomorphic connection on the associated vector bundle $\text{ad}(E_G)$.

Since $c_1(M) = 0$, a theorem due to Yau says that the holomorphic tangent bundle TM admits a Kähler–Einstein metric [16]. Fix a Kähler–Einstein Kähler form ω on M . The vector bundle TM is polystable because ω is Kähler–Einstein; see [10, p. 177, Theorem (8.3)] Hence from Lemma 2.1,

$$(3.6) \quad (c_2(TM) \cup \omega^{d-2}) \cap [M] \geq 0.$$

Since $\text{ad}(E_G)$ admits a holomorphic connection, and TM is polystable with

$$\text{degree}(TM) = 0,$$

the vector bundle $\text{ad}(E_G)$ is semistable (see [3, p. 2830]). Also, we have

$$(3.7) \quad c_j(\text{ad}(E_G)) = 0$$

for all $j \geq 1$ because $\text{ad}(E_G)$ admits a holomorphic connection [1, p. 192–193, Theorem 4].

Consider the short exact sequence of vector bundles in (2.2). Since $c_1(M) = 0$, from (3.5) and (3.7) it follows that

$$(3.8) \quad c_1(\text{ad}(E_H)) = 0.$$

Hence from the above observation that $\text{ad}(E_G)$ is semistable of degree zero we conclude that $\text{ad}(E_H)$ is also semistable (any subsheaf of $\text{ad}(E_H)$ violating the semistability condition of $\text{ad}(E_H)$ would also violate the semistability condition of $\text{ad}(E_G)$). Now from Lemma 2.1 and (3.7) we conclude that

$$(3.9) \quad (c_2(\text{ad}(E_H)) \cup \omega^{d-2}) \cap [M] \geq 0.$$

In view of (3.7), (3.8) and (3.9), from the short exact sequence in (3.5) it follows that

$$(c_2(TM) \cup \omega^{d-2}) \cap [M] = -(c_2(\text{ad}(E_H)) \cup \omega^{d-2}) \cap [M] \leq 0.$$

On the other hand, from Lemma 2.1,

$$(c_2(TM) \cup \omega^{d-2}) \cap [M] \geq 0.$$

Hence

$$(c_2(TM) \cup \omega^{d-2}) \cap [M] = 0.$$

Consequently,

$$((2d \cdot c_2(TM) - (d-1)c_1(TM)^2) \cup \omega^{d-2}) \cap [M] = 0.$$

Therefore, from Corollary 3 in [2, p. 40] we conclude that TM is projectively flat. Therefore,

$$\text{End}(TM) = TM \otimes \Omega_M^1$$

is a flat vector bundle. Hence $c_2(\text{End}(TM)) = 0$. But

$$c_2(\text{End}(TM)) = 2d \cdot c_2(TM) - (d-1)c_1(TM)^2.$$

Since $c_1(TM) = 0$, we conclude that $c_2(TM) = 0$. Now from Lemma 1 of [14] we know that M is holomorphically covered by a complex torus. This completes the proof of the theorem. \square

Remark 3.2. The holomorphic tangent bundle TM of a compact connected Kähler manifold M admits a holomorphic connection if and only if M admits an étale covering by a complex torus. Indeed, if $A \rightarrow M$ is a covering map, where A is a complex torus, then the flat connection on TA given by a trivialization of TA descends to a holomorphic flat connection on TM . To prove the converse, assume that TM admits a holomorphic connection. So $c_1(TM) = 0 = c_2(TM)$. Since TM is polystable [16], we know that TM is projectively flat [2, p. 40, Corollary 3]. Since $c_1(TM) = 0$, and TM is projectively

flat, it follows that the Kähler–Einstein metric on M is flat. Therefore, M is covered by a complex torus.

It should be pointed out that when TM admits a holomorphic connection, then exact sequence in (3.5) is not compatible with the holomorphic connection on $\text{ad}(E_G)$.

4. CARTAN GEOMETRIES AND RATIONAL CURVES

A smooth complex projective variety Z is said to be rationally connected if for any two given points of Z , there exists an irreducible rational curve on Z that contains them; see [12, p. 433, Theorem 2.1] and [12, p. 434, Definition–Remark 2.2] for other equivalent conditions. Note that a rationally connected projective manifold is connected.

Theorem 4.1. *Let (M, E_H, θ) be a holomorphic Cartan geometry of type G/H such that M is rationally connected. Then G/H is a rational homogeneous space, and $M \cong G/H$. Furthermore, (M, E_H, θ) is holomorphically isomorphic to the tautological parabolic geometry $(G/H, G, \theta_0)$.*

Proof. Let D^θ denote the holomorphic connection on the principal G -bundle E_G induced by θ (see Section 3). Since M is rationally connected, the curvature of D^θ vanishes (see [4, p. 160, Theorem 3.1]). From the fact that M is rationally connected it also follows that M is simply connected [6, p. 545, Theorem 3.5], [11, p. 362, Proposition 2.3].

Since D^θ is flat and M is simply connected, we have the developing map for the Cartan geometry (M, E_H, θ)

$$\gamma : M \longrightarrow G/H$$

which is a covering map. Since M is rationally connected, from the covering γ we know that G/H is also rationally connected. We saw above that this implies that G/H is simply connected. Hence γ is an isomorphism.

Since G/H is a simply connected homogeneous projective variety, we know that G/H is rational. This completes the proof of the theorem. \square

REFERENCES

- [1] M. F. Atiyah, Complex analytic connections in fibre bundles, *Trans. Amer. Math. Soc.* **85** (1957) 181–207.
- [2] S. Bando and Y.-T. Siu, Stable sheaves and Einstein-Hermitian metrics, in: *Geometry and analysis on complex manifolds*, pp. 39–50, World Sci. Publishing, River Edge, NJ, 1994.
- [3] I. Biswas, Vector bundles with holomorphic connection over a projective manifold with tangent bundle of nonnegative degree, *Proc. Amer. Math. Soc.* **126** (1998) 2827–2834.
- [4] I. Biswas, Principal bundle, parabolic bundle, and holomorphic connection, in: *A tribute to C. S. Seshadri (Chennai, 2002)*, 154–179, Trends Math., Birkhäuser, Basel, 2003.
- [5] I. Biswas, On parabolic geometry, II.
- [6] F. Campana, On twistor spaces of the class \mathcal{C} , *Jour. Diff. Geom.* **33** (1991) 541–549.
- [7] S. Dumitrescu, Structures géométriques holomorphes sur les variétés complexes compactes, *Ann. Sci. École Norm. Sup.* **34** (2001) 557–571.
- [8] S. Dumitrescu, Killing Fields of Holomorphic Cartan Geometries, <http://arxiv.org/abs/0902.2193>.

- [9] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, Aspects of Mathematics, E31. Friedr. Vieweg & Sohn, Braunschweig, 1997.
- [10] S. Kobayashi, *Differential geometry of complex vector bundles*, Princeton University Press, Princeton, NJ, Iwanami Shoten, Tokyo, 1987.
- [11] J. Kollár, Fundamental groups of rationally connected varieties, *Michigan Math. Jour.* **48** (2000) 359–368.
- [12] J. Kollár, Y. Miyaoka and S. Mori, Rationally connected varieties, *Jour. Alg. Geom.* **1** (1992) 429–448.
- [13] B. McKay, Rational curves and parabolic geometries, <http://arxiv.org/abs/math/0603276>.
- [14] B. McKay, Holomorphic parabolic geometries and Calabi-Yau manifolds, <http://arxiv.org/abs/0812.1749>.
- [15] R. W. Sharpe, *Differential Geometry*, Springer-Verlag, Heidelberg, 1997.
- [16] S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I, *Comm. Pure Appl. Math.* **31** (1978) 339–411.

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