LINEAR ALGEBRA PROBLEMS

You don't have to hand in these problems, but they could give you an idea of what to look for on the test.

(1) Suppose that $v_1, v_2, \dots, v_p$ are linearly independent vectors in a vector space V. Prove that $v_2 \neq 0$.

(2) Recall that a vector space V is said to be the *direct sum* of two subspaces U and W if every vector $v \in V$ admits a unique decomposition into a sum $u + w$ of some vector $u \in U$ and some vector $w \in W$. Symbolically, we denote a direct sum as $V = U \oplus W$. If $V = U \oplus W$ and V is finite dimensional then prove that
  (a) $\dim V = \dim U + \dim W$,
  (b) $U \cap W = \{0\}$.

(3) Suppose that V is a finite dimensional vector space and that U and W are subspaces of V. Prove that

$$\dim (U + W) = \dim U + \dim W - \dim (U \cap W) .$$

(Give a better proof than I gave in class.)

(4) The expression *direct sum* can be used differently. If V and W are two vector spaces, their direct sum, also written as $V \oplus W$ (confusingly) means the set of all pairs (v, w) for any $v \in V$ and $w \in W$. We add component-by-component, and scale likewise. Prove that

$$\dim (V \oplus W) = \dim V + \dim W.$$

(5) Suppose that $a < b$ are two real numbers, and that $x_1, x_2, \dots, x_n$ are distinct points of the interval $[a, b]$. Use linear algebra (and as little calculus as possible) to prove that there exist numbers $m_1, m_2, \dots, m_n$ so that, for any polynomial $p(x)$ of degree at most $n - 1$,

$$\int_a^b p(x)\, dx = m_1 p(x_1) + \dots + m_n p(x_n) .$$

(In other words, you can integrate by just making a finite sum.)

(6) Suppose that V is the space of all polynomials $p(x)$ of all degrees, and that

$$\alpha(p(x)) = p'(x)$$

and that

$$\beta(p(x)) = x\, p(x).$$

  (a) What is $\alpha \circ \beta$?
  (b) What is $\beta \circ \alpha$?

(7) Suppose that V and W are vector spaces. Let $\operatorname{Hom}(V, W)$ be the set of all linear maps $\phi\colon V \to W$. Add and scale linear maps in the obvious way to make $\operatorname{Hom}(V, W)$ into a vector space. Define a linear map

$$\operatorname{Hom}(\mathbb{R}^p, \mathbb{R}^q) \to \mathbb{R}^{pq},$$

and prove that your map is an isomorphism.

(8) Let V be the vector space of functions which are linear combinations of
$$1, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos(Nx), \sin(Nx).$$
Let $\phi\colon V \to V$ be the linear map
$$\phi(f) = \int_0^x f(u)\,du.$$
Calculate $\det \phi$.

(9) Say that a vector $v \in V$ is a *generalized eigenvector* of a linear map $\phi\colon V \to V$ with eigenvalue λ if there is some integer $k > 0$ so that
$$(\phi - \lambda I)^k\, v = 0.$$
For any fixed number $\lambda \in K$, the set of all generalized eigenvectors of ϕ with eigenvalue λ is called the generalized λ-eigenspace.
   (a) Prove that every eigenvector is a generalized eigenvector.
   (b) Prove that, for any $\lambda \in K$, and any linear map $\phi\colon V \to V$, the generalized λ-eigenspace of ϕ is a subspace.
   (c) Use the Jordan normal form of ϕ to prove that if V is a finite dimensional complex vector space, then V is a direct sum of the generalized eigenspaces of ϕ.

(10) Suppose that V is a real inner product space, and $v \in V$. The Schwarz inequality says that for any vectors $v, w \in V$,
$$|\langle v, w\rangle| \le |v|\;|w|\,.$$
Use the Schwarz inequality to prove that for every vector $v \in V$,
$$|v| = \max_{|w|=1} \langle v, w\rangle\,,$$
where the maximum is taken over all vectors $w \in V$ with $|w| = 1$.

(11) Suppose that V is a complex inner product space. Let $\mathrm{U}(V)$ be the set of all unitary linear maps $\phi\colon V \to V$. Prove that $\mathrm{U}(V)$ is a subgroup of $\mathrm{GL}(V)$.

(12) Find the center of $\mathrm{U}(V)$.

(13) Prove that for any $\phi \in \mathrm{U}(V)$, $\phi^{-1} = \phi^*$.

(14) Prove that for any $\phi in\, \mathrm{U}(V)$, $|\det \phi| = 1$.

(15) Suppose that V and W are finite dimensional vector spaces. Prove that there is a unique linear isomorphism
$$\phi\colon V \otimes W \to W \otimes V$$
so that
$$\phi\,(v \otimes w) = w \otimes v$$
for every $v \in V$ and $w \in W$.

(16) Suppose that V and W are finite dimensional vector spaces. To each tensor $\xi \otimes w \in V^* \otimes W$ associate the linear map $\phi\colon V \to W$ defined by
$$\phi(v) = \xi(v)w.$$
Prove that this association extends uniquely to a linear isomorphism
$$V^* \otimes W = \mathrm{Hom}\,(V, W)\,.$$