GROUP THEORY PROBLEMS

You don't have to hand in these problems, but they could give you an idea of what to look for on the test.

(1) Prove that any group has a unique identity element.
(2) Prove that every element of any group has a unique inverse.
(3) The symmetry group G of a square has eight elements. The quaternion group
$$H = \{1, i, j, k, -1, -i, -j, -k\}$$
(with $i^2 = j^2 = k^2 = -1$ and $ij = k, jk = i, ki = j, ji = -k, kj = -i$, and $ik = -j$, also has eight elements. Are they isomorphic? Prove your answer.
(4) A *cyclic group* is a group generated by a single element. Suppose that G is a cyclic group of order 14. Prove that G is isomorphic to $\mathbb{Z}_{14}$.
(5) If G is a group, an *automorphism* of G is an isomorphism $\phi\colon G \to G$. Prove that the set $\operatorname{Aut} G$ of all automorphisms of G is a group under composition.
(6) What is $\operatorname{Aut} \mathbb{Z}$?
(7) What is $\operatorname{Aut} \mathbb{Z}_4$?
(8) The *center* of a group G is the set $Z(G)$ consisting of all elements $a \in G$ so that, for every element $b \in G$, $ab = ba$. Find the center of the quaternion group.
(9) Prove that for any group G, $Z(G)$ is a normal subgroup of G.
(10) Find the center of
  (a) $\mathrm{GL}(2, \mathbb{R})$
  (b) $\mathrm{SL}(2, \mathbb{R})$
  (c) $\mathrm{GL}(2, \mathbb{C})$
  (d) $\mathrm{SL}(2, \mathbb{C})$.
(11) Suppose that $\phi\colon G \to L$ is a homomorphism of groups. Prove that
  (a)
$$\phi(1) = 1.$$
  (b) for any $g \in G$,
$$\phi\left(g^{-1}\right) = \phi(g)^{-1}.$$
(12) Prove that the kernel of any group homomorphism $\phi\colon G \to L$ is a normal subgroup of G.
(13) Give an example of a group homomorphism $\phi\colon G \to L$ whose image is not a normal subgroup of L.
(14) Write
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 6 & 9 & 7 & 2 & 5 & 8 & 1 & 3 \end{pmatrix}$$
as a product of disjoint cycles. What is the order of this permutation?
(15) Prove that every finite group of order 63 has a subgroup of order 7.