NOTES ON REPRESENTATIONS OF FINITE GROUPS

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Contents

1. Definitions

From here on, all vector spaces will be finite dimensional complex vector spaces. All groups will be finite groups.

Definition 1. Let V be a vector space. Then $\mathrm{GL}(V)$ is the the group of all isomorphisms from V to V.

Definition 2. A *representation* of a group G is a group morphism $\rho\colon G \to \mathrm{GL}(V)$.

So to each abstract group eleemnt $g \in G$, we associate a linear map $\rho(g)\colon V \to V$, so that for any $g, h \in G$, $\rho(gh) = \rho(g)\rho(h)$. In particular $\rho(1) = I$ and

$$\rho\left(g^{-1}\right) = \rho(g)^{-1}.$$

When the particular morphism ρ is clear, we will often say that V is a representation of G.

Definition 3. A *isomorphism* of two representations $\rho_1\colon G \to \mathrm{GL}(V_1)$ and $\rho_2\colon G \to \mathrm{GL}(V_2)$ is an isomorphism of vector spaces (just *one* linear map, not depending on the group element) say $\phi\colon V_1 \to V_1$ so that

$$\rho_2(g) \circ \phi = \phi \circ \rho_1(g),$$

for all $g \in G$.

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2. Examples

(1) Let G be the group of permutations of the numbers $1, 2, \dots, n$. Let V be an n-dimensional vector space with a basis $v_1, v_2, \dots, v_n$. Associate to each permutation p the linear map $\rho(p)$ given by
$$\rho(p)v_i = v_{p(i)}.$$
(2) Let G be the group $\mathbb{Z}_n$ and $V = \mathbb{C}$. Each linear map $V \to V$ is just multiplication by some complex number. Let
$$\rho(m) = e^{2\pi i m/n} \in \mathbb{C}^{\times}.$$
(3) For any group G, let $V = \mathbb{C}$, we can just let $\rho(g) = I$ for every $g \in G$. This ρ is called the *trivial* representation.

(4) Suppose that G is a finite group, and that V is a vector space whose dimension equals the number of elements in G. Suppose that V has a basis which is indexed by the elements of G, say
$$\{v_i\}_{i \in G}.$$
Then for each $g \in G$, define $\rho(g)$ by
$$\rho(g)v_i = v_{gi}.$$
This representation is called the *regular representation.*

(5) We can generalize the notion of representation.

Definition 4. If G is a group and X is a set, an *action* of G on X is a choice, for each $g \in G$, of permutation $p(g)\colon X \to X$, so that $p(gh) = p(g)p(h)$. If the choice of permutation $p(g)$ is understood, we will write $p(g)x$ as gx.

(6) For example, the group of permutations G of the numbers $1, 2, \dots, n$ acts in the obvious way on
$$X = \{1, 2, \dots, n\}.$$
(7) If G acts on X, take V to be any vector space with a basis
$$\{v_i\}_{i \in X}$$
indexed by the elements of the set X. Then let
$$\rho(g)v_x = v_{gx},$$
for each $x \in X$. We call ρ the *permutation representation* associated to the action on X.

(8) Recall that if V_1 and V_2 are two vector spaces, they have a *direct sum* $V_1 \oplus V_2$ consisting of the pairs (v_1, v_2) for $v_1 \in V_1$ and $v_2 \in V_2$, with pairwise addition and scaling. If $\rho_1\colon G \to \mathrm{GL}(V_1)$ and $\rho_1\colon G \to \mathrm{GL}(V_2)$ are two representations, then $V_1 \oplus V_2$ is a representation, via the rule
$$\rho(g)(v_1, v_2) = (\rho_1(g)v_1, \rho_2(g)v_2).$$
(9) Recall that if V_1 and V_2 are two vector spaces, they have a *tensor product* $V_1 \otimes V_2$. It is tricky to say what the elements of $V_1 \otimes V_2$ are, but we can say that if
$$u_1, u_2, \dots, u_p$$
is a basis of V_1 and
$$w_1, w_2, \dots, w_q$$

is a basis of V_2, then there are some elements

$$\begin{gathered} u_1 \otimes w_1, u_1 \otimes w_2, \dots, u_1 \otimes w_q, \\ u_2 \otimes w_1, u_2 \otimes w_2, \dots, u_2 \otimes w_q, \\ \vdots, \\ u_p \otimes w_1, u_p \otimes w_2, \dots, u_p \otimes w_q, \end{gathered}$$

forming a basis of $V_1 \otimes V_2$ (called the *product basis*). If $\rho_1 \colon G \to \mathrm{GL}\left(V_1\right)$ and $\rho_1 \colon G \to \mathrm{GL}\left(V_2\right)$ are two representations, then $V_1 \otimes V_2$ is a representation, via the rule

$$\rho(g)\left(u_i \otimes w_j\right) = \rho_1(g) u_i \otimes \rho_2(g) w_j.$$

3. Invariants

Definition 5. If $\rho \colon G \to \mathrm{GL}\,(V)$ is a representation, a *fixed vector* of the representation is a vector $v \in V$ so that $\rho(g)v = v$ for every $g \in G$. We denote by V^G the set of all fixed vectors. For any vector $v \in V$, we denote by $p(v)$ the vector

$$p(v) = \frac{1}{|G|} \sum_{g \in G} \rho(g) v,$$

the average of all of the things that G can do to v. We call p the *averaging operator*, and sometimes write $p(v)$ as v^G.

Lemma 1. *The averaging operator is linear, and G-invariant, and is a projection to V^G, i.e. $p \circ p = p$, and p is the identity on V^G.*

Proof. Linearity is clear. Clearly since we averaged over all of the different elements of G, p is G-invariant. If v is fixed, each $\rho(g)$ drops out, and we get $p(v) = v$. Clearly the $p(v)$ is G-invariant, i.e. the image of p is contained in V^G so $p \circ p = p$. ☐

4. Subrepresentations

Lemma 2. *Suppose that $\rho \colon G \to GL\,(V)$ is a representation of a finite group G. Take any inner product*

$$v, w \in V \mapsto \langle v, w \rangle \in \mathbb{C}.$$

From this inner product, define a new operation, which we will write as

$$v, w \in V \mapsto \langle v, w \rangle^G \in \mathbb{C},$$

and is defined by

$$\langle v, w \rangle^G = \frac{1}{|G|} \sum_{g \in G} \langle \rho(g) v, \rho(g) w \rangle .$$

(Average over all ways of getting G to act on the pair v, w of vectors). Then this new operation is a G-invariant inner product, i.e.

$$\langle \rho(g) v, \rho(g) w \rangle^G = \langle v, w \rangle^G ,$$

for any $g \in G$ and $v, w \in V$.

In particular, V has a G-invariant inner product.

Proof. Because we averaged over all the things that G can do, the result is clearly G-invariant. It is clearly linear in v and conjugate linear in w. We only need to check that $\langle v, v\rangle^G \geq 0$ and equals 0 only if $v = 0$. But $\langle v, v\rangle^G$ is a sum of nonnegative terms, vanishing only when $\rho(g)v = 0$ for all g. But $\rho(g)$ is linear isomorphism, so $\rho(g)v = 0$ just when $v = 0$. □

Example 1. If we let G be the group of permutations of $X = \{1, 2, \dots, n\}$, with the permutation representation on $V = \mathbb{C}^n$, then the usual inner product

$$\langle z, w\rangle = \sum_i z_i \bar{w}_i$$

is G-invariant.

Lemma 3. *If $\rho\colon G \to GL(V)$ is a representation of a finite group, then every element $\rho(g)$ is diagonalizable, and its eigenvalues are complex numbers of unit length.*

Proof. By lemma 2 on the preceding page, we can pick an invariant inner product. By definition, the inner product is invariant under every $\rho(g)$, so every $\rho(g)$ is unitary for that inner product. Unitary linear maps are normal, so unitarily diagonalizable. □

Warning: we can diagonalize each individual $\rho(g)$ in some basis, but maybe not all of them at once.

Definition 6. If V is a representation of G, then a *subrepresentation* of V is a G-invariant subspace of V.

Lemma 4. *If $\rho\colon G \to GL(V)$ is a representation of a finite group, then every subrepresentation has an invariant complement.*

Proof. If $W \subset V$ is a G-invariant subspace, then take any G-invariant inner product, and clearly $W^\perp$ is also G-invariant, and $V = W \oplus W^\perp$. □

Exercise 1. Suppose that G is a finite abelian group and $\rho\colon G \to \mathrm{GL}(V)$ is a representation. Prove that there is a basis, say

$$v_1, v_2, \dots, v_n,$$

so that *all* of the $\rho(g)$ are diagonal this one basis, i.e. for every g,

$$\rho(g)v_i = \lambda_i(g)v_i,$$

for some functions $\lambda_i\colon G \to \mathbb{C}^\times$.

Example 2. Let G be the symmetric group on 3 letters.

(1) One obvious representation is the trivial one, on $V = \mathbb{C}$.
(2) Consider the permutation representation of G on $V = \mathbb{C}^3$. Take the subspace $U \subset \mathbb{C}^3$ to be the vectors

$$z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

for which $z_1 + z_2 + z_3 = 0$. Clearly U is G-invariant.
(3) Define a representation of G on $V = \mathbb{C}$ by $\rho(g)z = \mathrm{sgn}(g)z$ (where sgn is the usual sign of a permutation).

Lemma 5. *Suppose that V is a representation of a finite group G, with averaging operator $p \colon V \to V$. Then $\operatorname{tr} p = \dim V^G$.*

Proof. The map p is a projection map (i.e. p is the identity on its image), and therefore we can split V into the kernel and image of p. On the kernel, $p = 0$, while on the image, $p = I$. □

5. Irreducible representations

Definition 7. A representation V of a group G is *irreducible* if it admits no G-invariant subspaces other than 0 and V.

Theorem 1. *Every representation of a finite group is a direct sum of irreducible representations.*

The proof is clearly by induction: take any subrepresentation W, and then V splits $V = W \oplus W^{\perp}$. Then try to split W and $W^{\perp}$, etc.

Exercise 2. Prove that every representation of any finite abelian group is a direct sum of dimension 1 representations.

Exercise 3. Prove that an irreducible representation of a finite group G has at most one G-invariant inner product.

Exercise 4. Prove that the 2-dimensional representation U of example 2 on the preceding page is irreducible.

6. Schur's lemma

Theorem 2. *Given two irreducible representations $\rho_1 \colon G \to GL(V_1)$ and $\rho_2 \colon G \to GL(V_2)$, any G-invariant linear map $\phi \colon V_1 \to V_2$ is either an isomorphism of representations or is* 0.

Either there is no isomorphism of representations from V_1 to V_2, or else there is an isomorphism and any two isomorphisms, say ϕ and ψ, satisfy $\psi = c\phi$ for some nonzero complex number c.

Proof. The kernel of ϕ is a G-invariant subspace of V_1, since ϕ is G-invariant. Therefore the kernel of ϕ is 0 or is V_1. The kernel is V_1 just when $\phi = 0$, so we can assume that the kernel of ϕ is 0. So ϕ is 1-1.

The image of ϕ is a G-invariant subspace of V_2, so either is 0 or is V_2. If the image of ϕ is 0 then $\phi = 0$, so we can assume that the image if V_2. But then ϕ is an isomorphism.

If ψ is another isomorphism, then let $\alpha = \psi^{-1} \circ \phi$. So α is a G-invariant linear isomorphism $V_1 \to V_1$. The eigenspaces of α are G-invariant, so each eigenspace is either V_1 or 0. So there can only be at most one eigenspace. Every linear map $V_1 \to V_1$ on any finite dimensional complex vector space has an eigenvalue, whose eigenspace must be nonzero, so must be V_1; let a be the corresponding eigenvalue. Then $\alpha = aI$, i.e. $\phi = a\psi$. □

If V is a vector space, it is traditional in representation theory to write $2V$ to mean $V \oplus V$, etc.

Theorem 3. *If V is a representation of a finite group G, then V splits into a sum of multiples of irreducible representations*

$$V = n_1 V_1 \oplus n_2 V_2 \oplus \cdots \oplus n_k V_k,$$

so that each V_i is not isomorphic to any V_j for $i \neq j$. The summands $n_i V_i$ are uniquely determined, as are the multiplicities n_i, up to ordering. The various V_i are uniquely determined up to isomorphism.

Proof. We know that we can split V into irreducible representations by theorem 1 on the preceding page. Suppose that we have two splittings as above, say

$$V = n_1 V_1 \oplus n_2 V_2 \oplus \dots \oplus n_k V_k, \qquad = p_1 W_1 \oplus p_2 W_2 \oplus \dots \oplus p_\ell W_\ell.$$

The identity map $V \to V$ yields a map on each factor V_i, and either identifies it with a factor W_j, or maps it to that factor by the 0 linear map. □

Example 3. Careful: take any group G and the trivial G representation on $V = \mathbb{C}^2$. Then we can split $\mathbb{C}^2 = \mathbb{C} \oplus \mathbb{C}$. But we could also (for example) let V_1 be the span of the vector

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and let V_2 be the span of the vector

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Then $V = V_1 \oplus V_2$. So V has many decompositions into G-invariant subspaces, but (by the theorem) only one into maximal sums of irreducible subspaces.

Definition 8. If $\rho_1 \colon G \to \mathrm{GL}(V_1)$ and $\rho_2 \colon G \to \mathrm{GL}(V_2)$ are two representations of a finite group, then $W = \mathrm{Hom}(V_1, V_2)$ (the set of all linear maps $V_1 \to V_2$) has G-representation

$$\rho \colon G \to \mathrm{GL}(W),$$

given by taking $\rho(g)\phi$ to be the linear map

$$(\rho(g)\phi)(v) = \rho_2(g)\phi\left(\rho_1(g)^{-1} v\right).$$

Exercise 5. Prove that this makes $\mathrm{Hom}(V_1, V_2)$ into a G-representation.

Definition 9. A *morphism of representations* is a linear map $\phi \colon V_1 \to V_2$ so that

$$\phi(\rho_1(g)v) = \rho_2 \phi(v),$$

for all $vinV_1$ and $g \in G$. Let $\mathrm{Hom}_G(V_1, V_2)$ be the set of all morphisms of representations $V_1 \to V_2$.

Exercise 6. Prove that $\mathrm{Hom}_G(V_1, V_2)$ is precisely the set of vectors in $\mathrm{Hom}(V_1, V_2)$ which are invariant under all elements of G, i.e.

$$\mathrm{Hom}_G(V_1, V_2) = \mathrm{Hom}(V_1, V_2)^G.$$

Lemma 6. *If V and W are two representations of a finite group G, say with each splitting into irreducibles as*

$$V = p_1 V_1 \oplus p_2 V_2 \oplus \dots p_k V_k,$$

and

$$W = q_1 W_1 \oplus q_2 W_2 \oplus \dots q_\ell W_\ell,$$

then $Hom_G(V, W)$ splits into irreducibles as

$$Hom_G(V, W) = \bigoplus_{ij} p_i q_j Hom(V_i, W_j),$$

where the direct sum is a sum over pairs i and j for which V_i and W_j are isomorphic. Moreover, Hom (V_i, W_j) is then 1-dimensional.

Proof. From theorem 3 on page 5, we have unique splittings into irreducibles. From theorem 2 on page 5, any linear map $\phi \in \text{Hom}(V, W)$ must split into a sum of isomorphisms and 0 maps. Moreover, each isomorphism is unique up to scaling, a 1-dimensional representation $\text{Hom}(V_i, W_j)$. □

7. Characters

Definition 10. The trace of a square (say $n \times n$) matrix A is

$$\operatorname{tr} A = A_{11} + A_{22} + \cdots + A_{nn},$$

the sum of the diagonal entries.

Exercise 7. Prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for any $n \times n$ matrices A and B.

Definition 11. The trace

$$\operatorname{tr} \phi$$

of a linear map $\phi \colon V \to V$ on a finite dimensional vector space V is defined by

$$\operatorname{tr} \phi = \operatorname{tr} A$$

where A is the matrix of ϕ in some basis.

Exercise 8. Prove that $\operatorname{tr} \phi$ does not depend on the choice of basis.

Choosing a basis in which A is in Jordan normal form, we see that $\operatorname{tr} \phi$ is the sum of the eigenvalues, counted with multiplicities.

Definition 12. If $\rho \colon G \to \text{GL}(V)$ is a representation, its *character* is the function χ (also written as χ_ρ) given by

$$\chi(g) = \operatorname{tr} \rho(g).$$

Exercise 9. Let G be the group of permutations of $1, 2, 3$. Write out each permutation, and find the value of the character of the permutation representation on each permutation.

Exercise 10. Prove that if G is the group of permutations of $1, 2, \dots, n$ then the character of the permutation representation is given by $\chi(g)$ equal to the number of numbers from among $1, 2, 3, \dots, n$ which are fixed by g.

Exercise 11. If χ is the character of an n-dimensional representation of a finite group G, then $\chi(1) = n$.

Lemma 7. *The average of the character of a representation is the dimension of the set of fixed vectors. In other words, if $\rho \colon G \to GLV$ is a representation with character χ, then*

$$\frac{1}{|G|} \sum_{g \in G} \chi(g) = \dim V^G.$$

Proof. Let $p \colon V \to V$ be the averaging operator. By lemma 5 on page 5, $\dim V^G = \operatorname{tr} p$. But

$$p(v) = \frac{1}{|G|} \sum_{g \in G} \rho(g) v,$$

i.e.

$$p = \frac{1}{|G|} \sum_{g \in G} \rho(g).$$

Thus

$$\begin{aligned}
\dim V^G &= \operatorname{tr} p \\
&= \frac{1}{|G|} \sum_{g \in G} \operatorname{tr} \rho(g) \\
&= \frac{1}{|G|} \sum_{g \in G} \chi(g).
\end{aligned}$$

□

Example 4. Let $\rho \colon G \to \mathrm{GL}(V)$ be the regular representation of a finite group G, and χ its character. Then $\chi(1) = |G|$. Moreover, exercise 10 on the preceding page tells us that $\chi(g)$ is the number of elements of G fixed by g. But if $g \neq 1$, then the action takes each $h \in G$ to $gh \in G$, and has no fixed points: if $gh = h$ then multiply on the right by h to find $g = 1$. So $\chi(g) = 0$ for $g \neq 1$. The average of χ is therefore 1. So there is a one dimensional space V^G of fixed vectors, which is precisely the span of the vector

$$\sum_{h \in G} v_h.$$

Lemma 8. *If χ is the character of an n-dimensional representation of a finite group G, then*

(1) $\chi\left(g^{-1}\right) = \bar{\chi}(g)$ *(complex conjugation), and*
(2) $\chi(gh) = \chi(hg)$,

for any $g, h \in G$.

Proof. The second property is exercise 7 on the previous page. The first is just that, for each g, the linear map $\rho(g)$ is unitarily and so is unitarily diagonalizable with eigenvalues being unit length complex numbers, say $\lambda_j = e^{i\theta_j}$. So

$$\begin{aligned}
\chi(g) &= \operatorname{tr} \rho(g) \\
&= \sum_j \lambda_j \\
&= \sum_j e^{i\theta_j}.
\end{aligned}$$

But then

$$\begin{aligned}
\chi\left(g^{-1}\right) &= \operatorname{tr} \rho\left(g^{-1}\right) \\
&= \operatorname{tr} \left(\rho(g)\right)^{-1} \\
&= \sum_j e^{-i\theta_j}.
\end{aligned}$$

□

Lemma 9. *If $\rho_1 \colon G \to GL(V_1)$ and $\rho_2 \colon G \to GL(V_2)$ are two representations of a finite group, with characters χ_1 and χ_2, then the character of $V_1 \oplus V_2$ is $\chi_1 + \chi_2$, and the character of $V_1 \otimes V_2$ is $\chi_1 \chi_2$.*

Proof. Fix one element $g \in G$. Pick a basis of eigenvectors

$$u_1, u_2, \dots, u_p \in V_1$$

of $\rho_1(g)$, say

$$\rho_1(g) u_i = \lambda_i u_i.$$

So these λ_i are the eigenvalues, and

$$\chi_1(g) = \sum_i \lambda_i.$$

Similarly, pick a basis of eigenvectors

$$w_1, w_2, \dots, w_q \in V_2$$

of $\rho_2(g)$, say

$$\rho_2(g) w_j = \mu_j w_j,$$

and

$$\chi_2(g) = \sum_j \mu_j.$$

Let $\rho(g)$ be the direct sum representation. Make the basis

$$(u_1, 0), (u_2, 0), \dots, (u_p, 0), (0, w_1), (0, w_2), \dots, (0, w_q) \in V_1 \oplus V_2.$$

These are eigenvectors of the representation on the direct sum, with eigenvalues

$$\lambda_1, \lambda_2, \dots, \lambda_p, \mu_1, \mu_2, \dots, \mu_q.$$

So the trace is the sum of the traces.

The proof for tensor products is similar. Clearly

$$\rho(g) u_i \otimes w_j = \lambda_i \mu_j u_i \otimes w_j,$$

so diagonal in the product basis, with character

$$\begin{aligned} \chi(g) &= \sum_{ij} \lambda_i \mu_j \\ &= \left(\sum_i \lambda_j \right) \left(\sum_j \mu_j \right) \\ &= \chi_1(g) \chi_2(g). \end{aligned}$$

□

Exercise 12. If $\rho \colon G \to \mathrm{GL}(V)$ is the permutation representation of some finite group G acting on some finite set X, and $\chi = \chi_\rho$, prove that, for any $g \in G$, $\chi(g)$ is the number of elements of X fixed by g.

Lemma 10. *If $\rho_1 \colon G \to GL(V_1)$ and $\rho_2 \colon G \to GL(V_2)$ are two representations of a finite group, with characters χ_1 and χ_2, then the character of the representation on Hom (V_1, V_2) is*

$$\chi = \chi_2 \bar{\chi}_1.$$

Proof. For each $g \in G$, again pick a basis $\{v_i\} \subset V_1$ of eigenvectors of $\rho_1(g)$,

$$\rho_1(g) v_i = \lambda_i v_i,$$

and a basis $\{w_j\} \subset V_2$ of eigenvectors of $\rho_2(g)$,

$$\rho_2(g) w_j = \mu_j w_j.$$

Then using a G-invariant inner product on V_1, we can define linear maps

$$\phi_{ij} \in \operatorname{Hom}(V_1, V_2)$$

by

$$\phi_{ij}(v) = \langle v, v_i \rangle w_j.$$

We then check that

$$\rho(g)\phi_{ij} = \mu_j \bar{\lambda}_i \phi_{ij}.$$

□

Exercise 13. Check the last step in the last proof.

8. Characters and class functions

Definition 13. A *class function* for a finite group G is a function $f \colon G \to \mathbb{C}$ so that $f(gh) = f(hg)$ for any $g, h \in G$.

For example, the character of any representation is a class function.

Exercise 14. Let C_G be the vector space of all class functions on G, with usual addition and scaling of functions. For $G = \mathbb{Z}_n$, what is the dimension of C_G?

Definition 14. We define an inner product on class functions. If f_1 and f_2 are two class functions, define the inner product

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \bar{f}_2(g),$$

(where the bar is complex conjugate).

Exercise 15. Prove that the space C_G of class functions is a finite dimensional inner product space, with this inner product.

Theorem 4. *If V_1 and V_2 are irreducible representations of a finite group G, say with characters χ_1 and χ_2, then*

$$\langle \chi_1, \chi_2 \rangle = \begin{cases} 1 & \text{if } V_1 \text{ is isomorphic to } V_2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let χ be the character of the representation $\operatorname{Hom}(V_1, V_2)$. Recall $\chi = \chi_2 \bar{\chi}_1$ from lemma 10 on the preceding page.

$$\begin{aligned} \langle \chi_1, \chi_2 \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_1(g) \bar{\chi}_2(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \chi(g) \end{aligned}$$

so by lemma 7,

$$= \dim \operatorname{Hom}(V_1, V_2)^G,$$

and by exercise 6 this is

$$= \dim \operatorname{Hom}_G (V_1, V_2),$$

and because V_1 and V_2 are irreducible this is

$$= \begin{cases} 1 & \text{if } V_1 \text{ is isomorphic to } V_2, \\ 0 & \text{otherwise.} \end{cases}$$

□

Theorem 5. *Suppose that V is a representation of a finite group G, with character χ, and V splits into a sum of multiples of irreducibles, say*

$$V = n_1 V_1 \oplus n_2 V_2 \oplus \cdots \oplus n_k V_k,$$

and V_i has character χ_i, then

$$n_i = \langle \chi, \chi_i \rangle .$$

In particular, two representations with the same character are isomorphic.

Corollary 1. *There are only finitely many irreducible representations of any finite group up to isomorphism.*

Proof. Each one gives us a unit vector in C_G, and all of the unit vectors given this way are orthonormal. But C_G has finite dimension. □

Theorem 6. *The character χ of a representation has $\langle \chi, \chi \rangle$ an integer, and this integer is 1 just when the representation is irreducible.*

Proof. Write out our representation as a sum of multiples of irreducibles, say

$$V = n_1 V_1 \oplus n_2 V_2 \oplus \cdots \oplus n_k V_k,$$

and so

$$\chi = n_1 \chi_1 + n_2 \chi_2 + \ldots n_k \chi_k,$$

if V_i has character χ_i. Then

$$\langle \chi, \chi \rangle = \sum n_i^2.$$

□

Theorem 7. *Every irreducible representation of a finite group appears as a summand in the regular representation. Its multiplicity equals its dimension.*

Proof. By exercise 12 on page 9, the character of the regular representation V has $\chi(g)$ equal to the number of fixed points of g acting on $X = G$. But g acts by taking $h \mapsto gh$. There are no fixed points unless $g = 1$, when every point is fixed. So

$$\chi(g) = \begin{cases} |G| & \text{if } g = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Take any character η of any irreducible representation W. Then the number n of times that W occurs in the regular representation V is

$$n = \langle \chi, \eta \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \bar{\eta}(g).$$

But $\chi(g) = 0$ except at $g = 1$, so

$$\begin{aligned} n &= \frac{1}{|G|} \chi(1) \bar{\eta}(1) \\ &= \frac{1}{|G|} |G| \bar{\eta}(1) \\ &= \bar{\eta}(1) \\ &= \dim W. \end{aligned}$$

□