Coláiste An Spioraid Naoimh Maths Circle
Lesson 10

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Last Week's Take-Home Problem

3 men checked into a hotel one day- the price was €30, so they each paid €10 and went to their room. The hotel manager then remembered that they had a special offer that night- €25 for a room. So he sent the bellboy up to the room with the €5 that he overcharged them- 5 €1 coins. The three men couldn't split the five coins between them so they each took €1 and gave the bellboy a €2 tip.

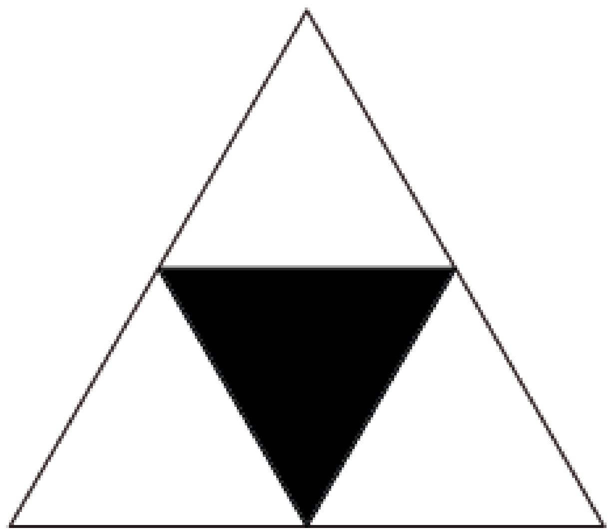
Because they each got €1 back, in fact, each of the men paid €9. $3 \times$ €9 = €27, adding on the €2 tip we get, €27 + €2 = €29. Where did the missing euro go???

Solution:

There is no reason why the tip added to the money they paid should equal the total money, €30. The money they paid should equal the cost of what they bought, i.e. the hotel room (€25) and the tip(€2). $3 \times$ €9 = €27 = €25 + €2.

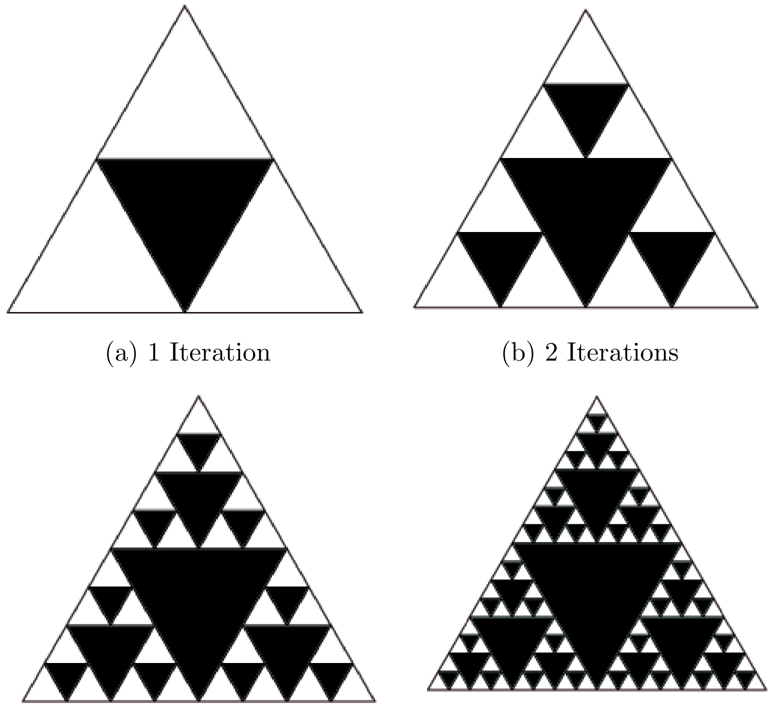
1. **Sierpinski's Triangle**

Take an equilateral triangle. Find the midpoint of each line and join these to form another triangle. Colour this new triangle, as shown:



What portion of the original triangle is left? Now do the same for the three smaller triangles that are left. How much of the original triangle will now be left. How much of the original triangle would be left if you do another iteration? And another? How about after n iterations? What about after an infinite amount of iterations?

Solution:



(a) 1 Iteration (b) 2 Iterations

(c) 3 Iterations (d) 4 Iterations

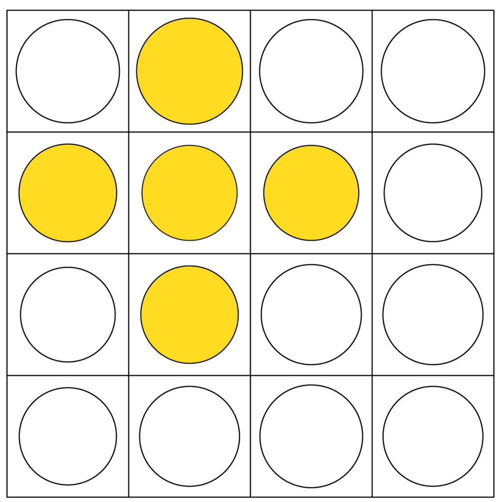
Figure 1: Iterations of Sierpinski's Triangle

After 1 iteration there is $\frac{3}{4}$ of the original triangle left. After the second iteration there will be $\frac{3}{4}$ of each of the 3 triangles left. So there will be $\frac{3}{4}$ of $\frac{3}{4} = \frac{9}{16}$ of the original triangle left. After n iterations there will be $(\frac{3}{4})^n$ left. After an infinite amount of iterations the area of the triangle will be 0.

The Sierpinski triangle is named after Polish mathematician Waclaw Sierpinski who described it in 1915. It is a fractal. Fractals are typically self simillar patterns, meaning that they look the same up close as they do far away. The shapes that we normally meet all have an integer dimension- e.g. a point is 0 dimensional, a line is 1 dimensional, a square is 2 dimensional, a cube is 3 dimensional, etc. Looking at the Sierpinski triangle, one may initially think that it is 2 dimensional. However if it is 2 dimensional, then why does it have an area of 0? Fractals actually have a fractional dimension. The dimension of Sierpinski's triangle is approximately 1.58.

2. Lights Out

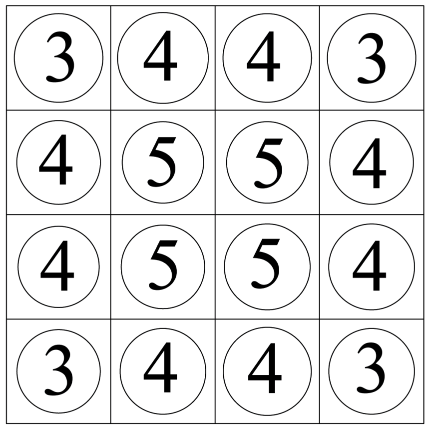
You are given 16 light bulbs in a 4×4 grid. The goal is to turn on all the light bulbs. The only problem is that any time you switch one bulb on or off, the state of all adjacent bulbs also changes:



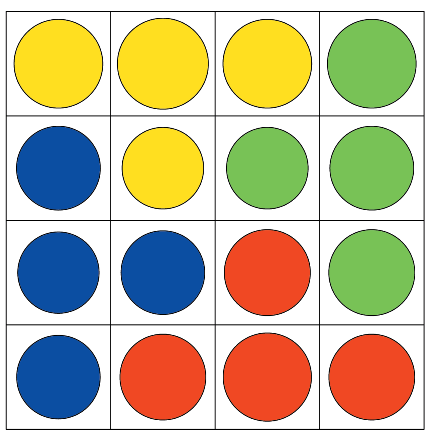
Above we see an example- the bulb in the second column, second row was switched and the bulbs directly above, directly below, to the left and to the right of it all turned on as well. The winner of the game is the person that can turn on all the bulbs in the fewest moves. What do you think is the least amount of moves that it can be done in?

Solution:

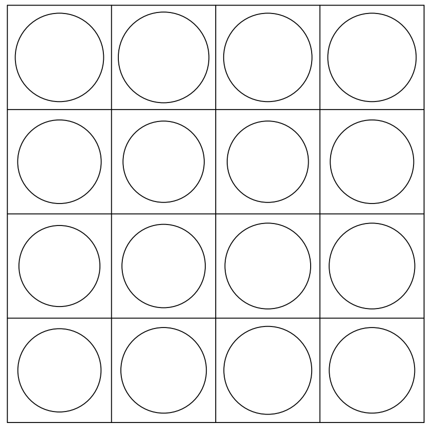
To think about what the least amount of moves it can be done in, we first look at the amount of bulbs that change state for each of the bulbs switched:



So each of the bulbs either change the state of 3, 4 or 5 bulbs. We want to turn on 16 bulbs. Since $5 + 5 + 5 = 15 < 16$, it can not be done in 3 or less moves. So we know that we need at least 4 moves. We know that $4+4+4+4 = 16$, so we may want to try 4 of the bulbs that change the state of exactly 4 bulbs:



Take Home Problem



Suppose we pressed each bulb in the lights out grid, one at a time- recall that each time we press a bulb it changes the state of that bulb, the bulbs directly above/below it and the bulbs to the right/left of it- what bulbs would be on/off at the end? Is the answer the same no matter what order you press the bulbs?