2010 IMO Summer IMO Training: Inequalities

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The most useful inequalities are the ones listed here:

QM-AM-GM-HM Inequality: Let $x_1, \cdots, x_n \geq 0$.

$$\sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}} \geq \frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n} \geq \frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}.$$

Don't forget about the *weighted* AM-GM inequality. Let $\alpha_1, \cdots, \alpha_n > 0$ and $\alpha = \alpha_1 + \cdots + \alpha_n$. Then

$$\frac{\alpha_1 x_1 + \cdots + \alpha_n x_n}{\alpha} \geq \sqrt[\alpha]{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}.$$

Cauchy Schwarz Inequality: Let $x_1, \cdots, x_n, y_1, \cdots, y_n \in \mathbb{R}$.

$$(x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2) \geq (x_1 y_1 + \cdots + x_n y_n)^2.$$

and the following corollaries,

$$x_1^2 + x_2^2 + \cdots + x_n^2 \geq \frac{(x_1 + \cdots + x_n)^2}{n}.$$

$$\frac{x_1^2}{y_1} + \cdots + \frac{x_n^2}{y_n} \geq \frac{(x_1 + \cdots + x_n)^2}{y_1 + \cdots + y_n}.$$

Jensen's Inequality: Let f be a convex function on a closed interval I and $x_1, x_2, \cdots, x_n \in I$. Let $a_1, \cdots, a_n$ be non-negative numbers such that $a_1 + a_2 + \cdots + a_n = 1$. Then

$$f(a_1 x_1 + a_2 x_2 + \cdots + a_n x_n) \leq a_1 f(x_1) + a_2 f(x_2) + \cdots + a_n f(x_n).$$

Of course, there is always the special case:

$$f\left(\frac{x_1 + \cdots + x_n}{n}\right) \leq \frac{f(x_1) + \cdots + f(x_n)}{n}.$$

If f is concave, then

$$f(a_1 x_1 + a_2 x_2 + \cdots + a_n x_n) \geq a_1 f(x_1) + a_2 f(x_2) + \cdots + a_n f(x_n).$$

Of course, there is always the special case:

$$f\left(\frac{x_1 + \cdots + x_n}{n}\right) \geq \frac{f(x_1) + \cdots + f(x_n)}{n}.$$

Muirhead's Inequality: Let $a_1, a_2, a_3, b_1, b_2, b_3 \geq 0$ such that $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$, $a_1 \geq a_2 \geq a_3$ and $b_1 \geq b_2 \geq b_3$. The triple (a_1, a_2, a_3) is said to *majorize* (b_1, b_2, b_3) if $a_1 \geq b_1$ and $a_1 + a_2 \geq b_1 + b_2$. Then for all $x, y, z \geq 0$,

$$\sum_{sym} x^{a_1} y^{a_2} z^{a_3} \geq \sum_{sym} x^{b_1} y^{b_2} z^{b_3}.$$

The following are the equality cases of Muirhead's Inequality:

1. If $(a,b,c)=(a',b',c')$, then clearly equality always holds.
2. Else if $c,c'>0$, then equality holds if and only if $x=y=z$ or at least one of x,y,z is 0.
3. Else if $c=c'=0$ and $b>0$, then equality holds if and only if $x=y=z$ or if two of x,y,z are equal and the third is equal to 0.
4. Otherwise, equality holds if and only if $x=y=z$.

Solving an inequality requires experience, trial and error. Sometimes, even if I tell you that the problem can be solved using AM-GM, it can be tricky to see how the AM-GM can be applied. It is not likely that the n terms to apply the AM-GM inequality will just be gift-wrapped and handed to you. [1]

Example 1: Let x,y,z be any positive real numbers such that $x+y+z=3$. Prove that

$$\frac{x^3}{(y+2z)^2}+\frac{y^3}{(z+2x)^2}+\frac{z^3}{(x+2y)^2}\geq\frac{1}{3}.$$

Proof: By AM-GM Inequality,

$$\frac{x^3}{(y+2z)^2}+\frac{y+2z}{27}+\frac{y+2z}{27}\geq 3x/9=x/3.$$

Therefore,

$$\frac{x^3}{(y+2z)^2}\geq\frac{9x-2y-4z}{27}.$$

Hence,

$$\sum_{cyc}\frac{x^3}{(y+2z)^2}\geq\sum_{cyc}\frac{9x-2y-4z}{27}=\frac{3(x+y+z)}{27}=\frac{1}{3},$$

as desired. $\square$

Why did I pick the denominator 27? The equality case of $AM-GM$ is when all terms applied are equal. Note that $x=y=z$ is an equality case of the whole inequality. Therefore, I want to use AM-GM to clear the denominator $(y+2z)$ in a way so that all three terms in $\frac{x^3}{(y+2z)^2}+\frac{y+2z}{27}+\frac{y+2z}{27}$ are equal. This occurs when the denominator of the latter two terms is 27.

Example 2: Let $x,y,z>0$. Find the minimum possible value of

$$\frac{x}{y}+\sqrt{\frac{y}{z}}+\sqrt[3]{\frac{z}{x}}.$$

Proof: Let $S=\frac{x}{y}+\sqrt{\frac{y}{z}}+\sqrt[3]{\frac{z}{x}}$. Note that

$$S=\frac{x}{y}+\frac{1}{2}\sqrt{\frac{y}{z}}+\frac{1}{2}\sqrt{\frac{y}{z}}+\frac{1}{3}\sqrt{\frac{z}{x}}+\frac{1}{3}\sqrt{\frac{z}{x}}+\frac{1}{3}\sqrt{\frac{z}{x}}.$$

[1] Seriously, do you expect your first problem to be proving $x+\frac{1}{x}\geq 2$ for all $x>0$?

Therefore,

$$S \geq 6\sqrt[6]{\frac{1}{2^2} \cdot \frac{1}{3^3} \cdot \frac{x}{y} \cdot \frac{y}{z} \cdot \frac{z}{x}} = \frac{6}{2^{1/3} \cdot 3^{1/2}}.$$

Equality holds when $\frac{x}{y} = \frac{1}{2} \cdot \sqrt{\frac{y}{z}} = \frac{1}{3}\sqrt{\frac{z}{x}}$. Some algebra yields that $x = 1, y = 2^{1/3}3^{1/2}, z = 3\sqrt{3}/2$ is an equality case. Therefore, $\frac{6}{2^{1/3} \cdot 3^{1/2}}$ is indeed the minimum of S. $\square$

Weighted AM-GM is very powerful when dealing with terms that are not symmetric and do not multiply "nicely".

Example 3: If a, b, c are positive real numbers such that

$$\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} = 2.$$

Prove that $ab + bc + ca \leq \frac{3}{2}$.

Solution: Rewriting the initial conditions yield

$$\sum_{cyc} \frac{a^2}{a^2+1} = 1.$$

By Cauchy Schwarz's Inequality, we have

$$\left(\sum_{cyc} \frac{a^2}{a^2+1}\right)\left(\sum_{cyc}(a^2+1)\right) \geq (a+b+c)^2.$$

Hence, $(a+b+c)^2 \leq a^2+b^2+c^2+3$. This is equivalent to $ab+bc+ca \leq \frac{3}{2}$. $\square$

It is important to interpret and use the initial conditions of an inequality in as many ways as possible to yield a solution to the inequality.

Example 4: Let a, b, c be non-negative real numbers such that $a + b + c = 1$. Prove that

$$a\sqrt{b} + b\sqrt{c} + c\sqrt{a} \leq \frac{1}{\sqrt{3}}.$$

Solution: Since $f(x) = \sqrt{x}$ is concave on $[0, \infty)$ and $a + b + c = 1$, by (weighted) Jensen's Inequality,

$$a\sqrt{b} + b\sqrt{c} + c\sqrt{a} \leq \sqrt{ab+bc+ca} \leq \sqrt{\frac{(a+b+c)^2}{3}} = \frac{1}{\sqrt{3}},$$

as desired. $\square$

The direct use of Jensen's inequality is straight forward. Applying a weighted version can be tricky.

Example 5: Let $a, b, c \geq 0$ such that $ab + bc + ca = 1$. Prove that

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{5}{2}.$$

Remember that other equality cases hold for Muirhead's inequality aside from $a = b = c$.

Solution Beginning: Assuming your strategy is using Muirhead's Inequality, you simply need to homogenize the inequality, i.e. make all the terms have the same degree. Since the left hand side has degree -1 and the right hand side has degree 0, and our initial condition has degree 2, we need to square both sides of our inequality before homogenizing. In the end, it suffices to prove that

$$(ab+bc+ca)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)^2 \geq \frac{25}{4}.$$

Now both sides of degree zero. It's bashing time. :)

Exercises:

1. Let x, y, z be positive real numbers such that $xyz = 1$. Prove that
$$\frac{x^3}{(1+y)(1+z)}+\frac{y^3}{(1+z)(1+x)}+\frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}.$$

2. Let $a, b, c > 0$ such that $abc \leq 1$. Prove that
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq a+b+c.$$

3. Let $a, b, c, d > 0$ such that $abcd = 1$ and $a+b+c+d > \frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}$. Prove that
$$a+b+c+d < \frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}.$$

4. Let $a, b, c \geq 0$ such that
$$\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1} = 2.$$
Prove that
$$\frac{1}{4a+1}+\frac{1}{4b+1}+\frac{1}{4c+1} \geq 1.$$

5. Let a, b, c be positive real numbers such that $ab+bc+ca \leq 3abc$. Prove that
$$\left(\sum_{cyc}\sqrt{\frac{a^2+b^2}{a+b}}\right)+3 \leq \sqrt{2}\left(\sum_{cyc}\sqrt{a+b}\right)$$

6. Let $a, b, c > 0$. Prove that
$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ca}}+\frac{c}{\sqrt{c^2+8ab}} \geq 1.$$

7. If a, b, c are positive real numbers such that $a+b+c = 3$, show that
$$\frac{1}{2+a^2+b^2}+\frac{1}{2+b^2+c^2}+\frac{1}{2+c^2+a^2} \leq \frac{3}{4}$$

8. Let a, b, c be non-negative real numbers such that $a+b+c=3$. Prove that

$$(a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2) \leq 12.$$

9. Let n be a positive integer and $x_1, x_2, \cdots, x_n \in \mathbb{R}$ such that $|x_i| \leq 1$ for each $i \in \{1, 2, \cdots, n\}$ and $x_1 + x_2 + \cdots + x_n = 0$.

   a.) Prove that there exists $k \in \{1, 2, \cdots, n\}$ such that

$$|x_1 + 2x_2 + \cdots + kx_k| \leq \frac{2k+1}{4}.$$

   b.) For $n > 2$, prove that the bound in (a) is the best possible. i.e. there exists $x_1, x_2, \cdots, x_n$ satisfying the initial conditions such that for all $k \in \{1, 2, \cdots, n\}$.

$$|x_1 + 2x_2 + \cdots + kx_k| \geq \frac{2k+1}{4}.$$

10. Let u, v, w be positive real numbers such that $u+v+w+\sqrt{uvw}=4$. Prove that

$$\sqrt{\frac{uv}{w}} + \sqrt{\frac{vw}{u}} + \sqrt{\frac{wu}{v}} \geq u+v+w.$$

Hints

1. Use the same technique as Example 1.
2. Try to use weighted AM-GM inequality. Can you assume that $abc = 1$?
3. Try to use weighted AM-GM inequality.
4. Let $(u, v, w) = (\frac{1}{a+1}, \frac{1}{b+1}, \frac{1}{c+1})$. Then apply Jensen's inequality on a natural function.
5. Apply inequalites to the right-hand side to result in something that look close to the left-hand side. Use QM-AM.
6. Use the same technique as Example 4.
7. Use the same technique as Example 3, plus a few more clever inequalities.
8. WLOG, suppose $a \geq b \geq c$. Even an inequality as weak as $b^2 - bc + c^2 = b^2 - c(b - c) \leq b^2$ can solve this inequality.
9. (a) Suppose there exists a minimum k where the inequality fails. (b) should not be too hard. Consider the cases when n is odd and even separately.
10. Remember that if $u, v, w \geq 0$, then $u^2 + v^2 + w^2 + 2uvw = 1$ if and only if $u = \cos A, v = \cos B, w = \cos C$ for some triangle ABC.