Parallel lines

(1) Let ABC be a triangle with $|AB| = |AC|$. Let D be an interior point of the side BC. The perpendicular from the point D to the line BC intersects the lines AB and AC in the points E and F. Show that the sum $|DE| + |DF|$ remains constant as D moves along BC.

(2) Let ABC be a triangle with $|AB| = |AC|$. Let D be an interior point of the side BC. The perpendicular from the point D to the line AB intersects AB at the point E and the perpendicular from D to AC intersects AC at the point F. Show that the sum $|DE| + |DF|$ remains constant as D moves along BC.

(3) Let ABC be a triangle. Let D be an interior point of the side AB, and E of AC such that $|BD| = |CE|$. Let M and N be the midpoints of the segments DE and BC, respectively. Prove that the line MN is parallel to the bisector of the angle $\widehat{BAC}$.

[Hint: Draw MB' parallel and equal to BD, and MC' parallel and equal to CE. We call that translating BD and CE so they pass through M.]

(4) * Let $ABCD$ be a convex quadrilateral and let M, N, P, and Q be interior points of the sides AB, BC, CD and DA respectively, such that the following relations hold:

$$\frac{AM}{MB} = \frac{DP}{PC} = a \text{ and } \frac{BN}{NC} = \frac{AQ}{QD} = b.$$

Let O be the point of intersection of the lines MP and NQ. Show that the following relations hold:

$$\frac{QO}{ON} = a \text{ and } \frac{MO}{OP} = b.$$

[Translate AB and CD so that they pass through Q.]

(5) Let ABC and $AB'C'$ be two triangles such that

$$\frac{AB}{AB'} = \frac{AC}{AC'} = \frac{BC}{B'C'}$$

and such that AC lies in the interior of the angle $\widehat{BAB'}$, while AB' lies in the interior of the angle $\widehat{CAC'}$. Prove that

a) the angle between the lines BC and $B'C'$ is equal to that between AB and AB'.

b) If M is an interior point of BB' and N is an interior point of CC' such that

$$\frac{MB}{MB'} = \frac{NC}{NC'}$$

show that the triangle AMN is similar to ABC.

(6) * Let ABC and $A'B'C'$ be two triangles such that

$$\widehat{BAC} = \widehat{B'A'C'}, \quad \widehat{ACB} = \widehat{A'C'B'}, \quad \widehat{CBA} = \widehat{C'B'A'}$$

(here the angles are measured clockwise, such that, for example, $\widehat{BAC} = -\widehat{CAB}$). Let A'', B'' and C'' be interior points of the segments AA', BB' and CC' respectively, such that

$$\frac{AA''}{AA'} = \frac{BB''}{BB'} = \frac{CC''}{CC'}.$$

Show that the triangle $A''B''C''$ is similar to the triangle ABC. [Translate ABC and $A'B'C'$ so that they all pass through A.]

(7) Use Menelaus' theorem to prove that the centroid of a triangle divides each median in the ratio $[2 : 1]$.

(8) Let $BMCP$ be a trapezoid with $BM \parallel CP$, and let A be the point of intersection of the lines BP and MC. Let N be the point of intersection of lines MP and CB. Knowing that $|AM| = 3$ cm and $|AC| = 7$ cm, calculate $\frac{|CN|}{|NB|}$.

Let B' be a point on the line BM and P' a point on the line PC such that A, B' and P' are collinear. Let N' be the point of intersection of lines MP' and CB'. Prove that NN' is parallel to BB'.

(9) (**Ceva's Theorem**) Consider $\triangle ABC$ and P a point in its interior. Let A_1, B_1, C_1 be the intersections of PA with BC, PB with CA, PC with AB, respectively.

a) Use Menelaus' Theorem in $\triangle ABA_1$ and $\triangle ACA_1$ to prove Ceva's formula:

$$\frac{|AC_1|}{|C_1B|} \cdot \frac{|BA_1|}{|A_1C|} \cdot \frac{|CB_1|}{|B_1A|} = 1$$

b) Reciprocally, if the formula above holds for three random points A_1, B_1, C_1 on the three sides of the triangle, prove that the lines AA_1, BB_1 and CC_1 intersect at the same point P.

(10) Consider a quadrilateral $ABCD$. Let M denote the intersection point of lines AB and CD, and let N denote the intersection point of lines BC and AD. Let P be the midpoint of the diagonal AC, let Q be the midpoint of diagonal BD, let R be the midpoint of MN, let X be the midpoint of DN, let Y be the midpoint of CN and Z the midpoint of CD.

a) Prove that the points X, Z and Q are collinear. Also prove that X, Y and R are collinear and that Y, Z and P are collinear.

b) Use Menelaus' Theorem in $\triangle XYZ$ to prove that the points P, Q and R are collinear.

(11) Let ABC be a triangle and P a point in its interior, not lying on any of the medians of ABC. Let A_1, B_1, C_1 be the intersections of PA with BC, PB with CA, PC with AB, respectively, and let A_2, B_2, C_2 be the intersections of B_1C_1 with BC, C_1A_1 with CA, A_1B_1 with AB, respectively.

a) Prove that $\frac{|BA_1|}{|A_1C|} = \frac{|BA_2|}{|A_2C|}$. Find similar relations on the lines AB and AC.

b) Use a) above to prove that the points A_2, B_2, C_2 are collinear.

c) * Prove that the midpoints of A_1A_2, B_1B_2, C_1C_2 are collinear.

(12) Draw a large parallelogram. Trace the contours of some coins to draw some circles of various sizes. Some circles will intersect all sides of your parallelogram, others are so small that they can fit completely inside it. Using only an unmarket ruler, find the centres of the circles.

Cyclic quadrilaterals and antiparallel lines

(1) Let ABC be a triangle. Consider three points A', B' and C' on BC, AC and AB, respectively. The circumcircles of $\triangle AB'C'$ and $CA'B'$ intersect at two points B' and S. Prove that $BA'SC'$ is a cyclic quadrilateral.

(2) Prove that a parallelogram is cyclic if and only if it is a rectangle. Prove that a trapezoid (trapezium) is cyclic if and only if it is an isosceles trapezoid.

(3) Let ABC be a triangle. Consider two points B' and C' on AB and AC, respectively. Show that BC and $B'C'$ are parallel if and only if the circumcircles of ABC and $AB'C'$ are tangent to each other. [Hint: two circles are tangent to each other at A if they have a common tangent at A.]

(4) Let ABC be a triangle with sides $AB = c$, $AC = b$ and $BC = a$. Let R be the radius of its circumcircle and let AA' be a diameter of the circumcircle. Prove that $\widehat{ABC} = \widehat{AA'C}$ and hence prove that

$$\frac{a}{\sin \hat{A}} = \frac{b}{\sin \hat{B}} = \frac{c}{\sin \hat{C}} = 2R.$$

(5) Let $ABCD$ be a convex quadrilateral, let E be the point at the intersection of the lines AD and BC and F the point at the intersection of the lines AB and CD.
a) Show that the circumcircles of the triangles EDC, EAB, FBC and FAD all intersect at a point M (Monge's point).
b) Prove that M is on the same circle with all the circumcentres of the circles above.

(6) * All the Monge's points for the quadrilateral in a pentagonal star are on a circle.

(7) a) Let H be the orthocentre of $\triangle ABC$. Let $AH \bigcap BC = \{D\}$, $BH \bigcap AC = \{E\}$, $CH \bigcap AB = \{F\}$.
a) Prove that $AEHF$ and $BCEF$ are cyclic quadrilaterals. Find four other cyclic quadrilaterals in the diagram.
b) Find all possible pairs of similar triangles in diagram.
c) Let M, N, P be the midpoints of the sides BC, CA and AB respectively, and M', N', P' the midpoints of AH, BH and CH. Prove that $MNM'N'$ is a rectangle. Find two other rectangles in the picture. Prove that $DMNP$ is an isosceles trapezoid. Find as many isosceles trapezoids in the picture as possible.
d) Prove that M, N, P together with D, E, F, and M', N', P' all lie on the same circle. This is called the 9-point circle, or Euler's circle.

(8) (Internal and external angles) Let $\mathcal{C}$ be a circle of center O and P a point in the interior of $\mathcal{C}$. Let A, B, C, D be points on $\mathcal{C}$ such that P is inside the segments AB and CD. Let Q be the intersection point of the lines AD and BC. Prove that

$$\widehat{APC} = \frac{1}{2}(\widehat{AOC} + \widehat{BOD}) \text{ and } \widehat{AQC} = \pm\frac{1}{2}(\widehat{AOC} - \widehat{BOD}).$$

(9) Let P be a point not on a circle. Two lines passing through P intersect the circle $\mathcal{C}$ of center O and radius R at the points A and B, respectively A' and B'. Prove that $\triangle PAA' \sim \triangle PB'B$. Hence prove that

$$PA \cdot PB = PA' \cdot PB' = |PO^2 - R^2|.$$

(10) * (IMO 2004) In a convex quadrilateral $ABCD$ the diagonal BD does not bisect the angles ABC and CDA. The point P lies inside $ABCD$ and satisfies

$$\widehat{PBC} = \widehat{DBA} \text{ and } \widehat{PDC} = \widehat{BDA}.$$

Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

(11) * (IMO 2008) An acute angled triangle has orthocentre H. The circle passing through H with centre the midpoint of BC intersects the line BC at A_1 and A_2. The circle passing through H with centre the midpoint of AC intersects AC at B_1 and B_2, and the circle passing through H with centre the midpoint of AB intersects AB at C_1 and C_2. Show that $A_1, A_2, B_1, B_2, C_1, C_2$ lie on a circle.

(12) * Let ABC be a triangle, let I be the incentre, let I_A, I_B and I_C be the centers of the circles externally tangent to sides. Let A', B', C' be the feet of the perpendiculars from I to the sides of the triangle, and A'', B'', C'' the feet of the perpendiculars from I_A to BC, from I_B to AC and from I_C to AB, respectively.

a) Show that I_AA, IBB, I_CC are the altitudes of $I_AI_BI_C$.

b) Show that I_AA'', I_BB'', I_CC'' intersect at the circumcircle O'' of $I_AI_BI_C$, of radius $O''I_A = 2R$, where R is the circumradius of ABC. (Hint: use 7)

c) Find all pairs of similar triangles in the figure. In particular, prove that

$$\frac{A'B'}{I_AI_B} = \frac{B'C'}{I_BI_C} = \frac{C'A'}{I_CI_A} = \frac{r}{2R},$$

where r is the inradius of ABC.

d) Show that I_AA', I_BB', I_CC' intersect at a point S. Furthermore, S is collinear with I, O'', the orthocentre of $A'B'C'$ and the centroids of $I_AI_BI_B$ and $A'B'C'$.

e) Let M, N, P be the midpoints of the sides BC, CA and AB respectively. The perpendicular from M to I_BI_C, from N to I_CI_A and from P to I_AI_B intersect at a point Q, such that Q, I and G are collinear, where G is the centroid of ABC, and $IG = 2QG$.

(13) * (Ptolemy's relation) Show that $ABCD$ is a cyclic quadrilateral if and only if

$$AB \cdot CD + AD \cdot BC = AC \cdot BD.$$

(14) * Let ABC be a triangle with $AB = AC = BC$ and let P be a point on its circumcircle. Show that the sum

$$PA + PB + PC$$

does not depend on the position of P on the circumcircle.

(15) * (IMO 2007) In triangle ABC the bisector of the angle BCA intersects the circumcircle again at R, the perpendicular bisector of BC at P, and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles RPK and RQL have the same area.

(16) * Let $ABCD$ be a cyclic quadrilateral and let P be the intersection point of its diagonals. Show that there exists a point Q such that the triangles PBC and QBC are similar, and so are triangles PAD and QAD.

(17) * Let ABC and $A'B'C'$ be similar to each other as in question 6 (such that the orientation of the angles counts, too). Choose points A'', B'' and C'' such that the triangles $AA'A''$, $BB'B''$ and $CC'C''$ are all similar to each other, (again, the orientation of the angles counts). Guess what?

(Hint: find a "centre of similarity", like in question 16).

(18) * (IMO 2009) Let ABC be a triangle with circumcentre O. The points P and Q are interior points of the sides CA and AB, respectively. Let K, L, and M be the midpoints of the segments BP, CQ and PQ, respectively, and let Γ be the circle passing through K, L, and M. Suppose that the line PQ is tangent to the circle Γ. Prove that OP=OQ.

Anca Mustata, School of Mathematical Sciences, UCC

E-mail address: a.mustata@ucc.ie