EUCLIDEAN GEOMETRY

ANCA MUSTATA

1. Congruent triangles

We recall briefly the basics of congruent triangles.

Theorem 1.1. ***(Side-Angle-Side or SAS)*** *If two triangles $\triangle ABC$ and $\triangle A'B'C'$ satisfy*

$$\left.\begin{array}{c} |AB| = |A'B'| \\ \hat{A} = \hat{A'} \\ |AC| = |A'C'| \end{array}\right\} \text{ then } \triangle ABC \equiv \triangle A'B'C'.$$

Theorem 1.2. ***(Angle-Side-Angle or ASA)*** *If two triangles $\triangle ABC$ and $\triangle A'B'C'$ satisfy*

$$\left.\begin{array}{c} \hat{A} = \hat{A'} \\ |AB| = |A'B'| \\ \hat{B} = \hat{B'} \end{array}\right\} \text{ then } \triangle ABC \equiv \triangle A'B'C'.$$

As an application we have:

Theorem 1.3. *Consider a triangle ABC. The following two statements are equivalent:*

(1) *$\triangle ABC$ is isosceles with $|AB| = |AC|$.*
(2) *$\hat{B} = \hat{C}$.*

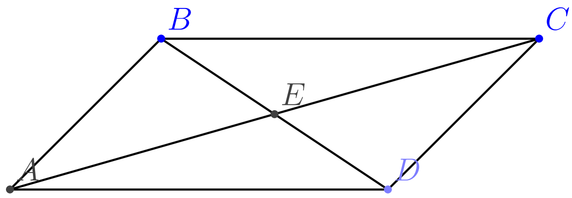
Theorem 1.4. ***(Side-Side-Side or SSS)*** *If two triangles $\triangle ABC$ and $\triangle A'B'C'$ satisfy*

$$\left.\begin{array}{c} |AB| = |A'B'| \\ |AC| = |A'C'| \\ |BC| = |B'C'| \end{array}\right\} \text{ then } \triangle ABC \equiv \triangle A'B'C'.$$

2. Applications of congruent triangle criteria to parallelograms

Definition 2.1. A quadrilateral whose opposite sides are parallel is called a parallelogram.

Theorem 2.2. *Let $ABCD$ be a quadrilateral. The following statements are equivalent:*
(A) $ABCD$ is a parallelogram.
(B) $AB \parallel CD$ and $|AB| = |CD|$.
(C) The diagonals AC and BD intersect at their midpoint.
(D) $|AB| = |CD|$ and $|BC| = |AD|$.



Proof. The four statements (A)-(D) are equivalent if each of them implies any other among them. Thus it seems that we would have to prove a total of 12 implications! (check that 12 is the correct number). However, we can considerably shorten our work by following a diagram like this one:

$$\begin{tikzcd}
(A) \arrow[r, "\text{Part I}"] & (B) \arrow[d, "\text{Part II}"] \\
(D) \arrow[u, "\text{Part IV}"] & (C). \arrow[l, "\text{Part III}"']
\end{tikzcd}$$

Note that we can connect any two points in the diagram by a sequence of arrows. Other strategies (arrow diagrams) can be designed for the same problem, but this is one of the shortest with just 4 implications.

Part I We assume (A) and prove (B).

We only need to prove $|AB| = |CD|$.

$$\left.\begin{array}{r} AB \parallel CD \Rightarrow \widehat{ABD} = \widehat{CDB} \\ |BD| = |DB| \\ BC \parallel AD \Rightarrow \widehat{BDA} = \widehat{DBC} \end{array}\right\} \quad (A.S.A.) \Rightarrow \triangle ABD \equiv \triangle CDB$$

$$\Rightarrow |AB| = |CD|.$$

Part II We assume (B) and prove (C). Let E denote the intersection point of the diagonals.

$$\left.\begin{array}{r} \widehat{ABD} = \widehat{CDB}(\text{alternate angles for } AB \| CD) \\ |AB| = |CD| \\ \widehat{BAC} = \widehat{DCA}(\text{alternate angles for } AB \| CD) \end{array}\right\} \text{ then} \qquad (A.S.A.) \Rightarrow \triangle ABE \equiv \triangle CDE$$

$$\Rightarrow |AE| = |CE| \text{ and } |BE| = |DE|.$$

Part III We assume (C) and prove (D).

$$\left.\begin{array}{r} |AE| = |CE| \\ \text{Opposite angles } \widehat{BEA} = \widehat{DEC} \\ |BE| = |DE| \end{array}\right\} \text{ then} \quad (S.A.S.) \Rightarrow \triangle ABE \equiv \triangle CDE$$

$$\Rightarrow |AB| = |CD|$$

Similarly, we can prove $\triangle CBE \equiv \triangle ADE$ and hence $|CB| = |AD|$.

Part IV We assume (D) and prove (A).

$$\left.\begin{array}{r} |AB| = |CD| \\ |BC| = |AD| \\ |AC| = |AC| \end{array}\right\} \text{then} \qquad (S.S.S.) \Rightarrow \triangle ABC \equiv \triangle CDA$$

$$\Rightarrow \widehat{BAC} = \widehat{DCA} \text{ and } \widehat{ACB} = \widehat{CAD}.$$

$$\Rightarrow AB \parallel CD \text{ and } BC \parallel AD \text{ (by the theorem on parallel lines).}$$

□

2.1. **Special parallelograms.**

Definition 2.3. A quadrilateral with four right angles is called a rectangle.

We note that a rectangle is a parallelogram by the Theorem on parallel lines (**??**).

Theorem 2.4. *Let $ABCD$ be a quadrilateral. The following statements are equivalent:*
(A) $ABCD$ is a rectangle.
(B) The diagonals AC and BD intersect at their midpoint and $|AC| = |BD|$.

Proof. **Part I** We assume (A) and prove (B).

$$\left.\begin{array}{r} \text{By the theorem on parallelograms, } |AB| = |CD| \\ \widehat{BAD} = \widehat{CDA} = 90^\circ \\ |AD| = |DA| \end{array}\right\} \quad (S.A.S.) \Rightarrow \triangle ABD \equiv \triangle DCA$$

$$\Rightarrow |AC| = |BD|.$$

Part II We assume (B) and prove (A). We first note that $ABCD$ is a parallelogram by the Theorem on parallelograms (D), since the diagonals AC and BD intersect at their midpoint.

$$\left.\begin{array}{r} \text{By the theorem on parallelograms, } |AB| = |CD| \\ \text{By assumption, } |AC| = |BD| \\ |AD| = |DA| \end{array}\right\} \quad (S.S.S.) \Rightarrow \triangle ABD \equiv \triangle DCA$$

$$\Rightarrow \widehat{BAD} = \widehat{CDA}$$

and, since the sum of the two angles is 180° by the Theorem on parallel lines, it follows that $\widehat{BAD} = \widehat{CDA} = 90^\circ$. Then again, by the Theorem on parallel lines, $\widehat{BAD} + \widehat{ABC} = 180^\circ$ and $\widehat{CDA} + \widehat{DCB} = 180^\circ$, so $\widehat{ABC} = \widehat{DCB} = 90^\circ$. □

Definition 2.5. A quadrilateral all of whose sides are equal is called a rhombus.

We note that any rhombus is a parallelogram by the theorem on parallelograms (C).

Theorem 2.6. *Let $ABCD$ be a quadrilateral. Show that the following statements are equivalent:*
(A) $ABCD$ is a rhombus.
(B) The diagonals AC and BD intersect at their midpoint and $AC \perp BD$.

Proof. **Part I** We assume (A) and prove (B).

$\triangle BAC$ and $\triangle DAC$ are isosceles with $|AB| = |BC| = |CD| = |DA|$

$$\Rightarrow \widehat{BAC} = \widehat{BCA} = \widehat{DAC} = \widehat{DCA} = \frac{1}{2}\widehat{BAD}. \tag{2.1}$$

Similarly, $\triangle ABD$ and $\triangle CBD$ are isosceles with $|AB| = |BC| = |CD| = |DA|$

$$\Rightarrow \widehat{ABD} = \widehat{ADB} = \widehat{CBD} = \widehat{CDB} = \frac{1}{2}\widehat{ADC}. \tag{2.2}$$

On the other hand,

$$AB \parallel CD \Rightarrow \widehat{BAD} + \widehat{ADC} = 180°.$$

This, together with equations (2.1) and (2.2), implies

$$\widehat{BAC} + \widehat{ABD} = 90°$$

□

Part II We assume (B) and prove (A).
Let E be the intersection point of AC and BD.

$$\left.\begin{array}{r}\text{By assumption, } |AE| = |CE| \\ \text{By assumption, } \widehat{BEA} = \widehat{BEC} = 90° \\ |BE| = |BE|\end{array}\right\} \text{ then } \quad (S.A.S.) \Rightarrow \triangle ABE \equiv \triangle CBE$$

$$\Rightarrow |AB| = |BC|.$$

Then by the Theorem on parallelograms, all sides are equal.

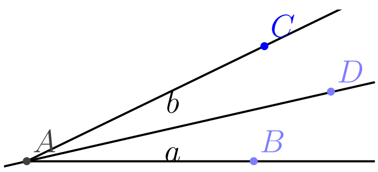
3. Special lines in a triangle

In this section we will use congruence of triangles to study five types of important lines in a triangle: the angle bisectors, the perpendicular bisectors, the altitudes and the medians, as well as the midlines. We will prove that the three angle bisectors of a triangle intersect at a unique point. Similarly for the three perpendicular bisectors; for the three altitudes and for the three medians. We will also study the defining properties of the points forming these lines.

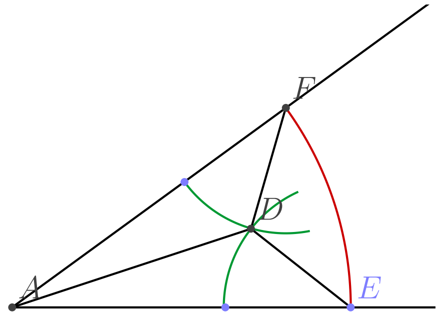
3.1. **Angle bisector.**

Definition 3.1. The angle bisector of an angle $\widehat{BAC}$ is the line AD such that D is a point in the interior of the angle and $\widehat{BAC}$

$$\widehat{BAD} = \widehat{DAC}.$$



Construction of the angle bisector. Draw an arc of circle centered at A, and let the intersection points with the rays AB and AC be E and F, respectively. Two arcs of circles centered at E and F respectively, and of the same radius, will intersect at a point D.



Then AD is the angle bisector of $\widehat{BAC}$. Indeed,

$$\left.\begin{array}{r}\text{By construction}|AE| = |AF| \\ \text{By construction, } |ED| = |FD| \\ |AD| = |AD|\end{array}\right\} \quad (S.S.S.) \Rightarrow \triangle AED \equiv \triangle AFD$$

$$\Rightarrow \widehat{EAD} = \widehat{DAF}.$$

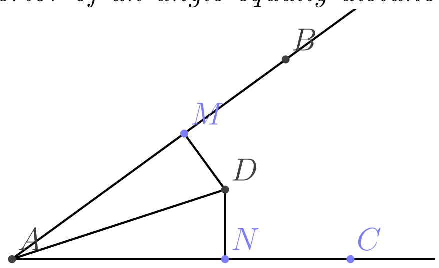
There is an alternative way of characterizing the bisector, involving the distance from a point to a line.

Definition 3.2. The distance from a point D to a line AB not containing the point is the length $|DM|$, where $DM \perp AB$ and $M \in AB$. Note that this is the same as the shortest path from D to a point on AB. (due to Pythgoras' theorem.)

Definition 3.3. A geometric locus is by definition a set of points in plane all of which satisfy a given property.

Theorem 3.4. ***The universal property of an angle bisector (or, in other words, the angle bisector described as a geometric locus.)*** *Consider an angle $\widehat{BAC}$ and a point D in its interior. The following two statements are equivalent:*
(A) AD is the angle bisector of $\widehat{BAC}$.
(B) The point D is equally distanced from AB and AC.

In other words, the angle bisector is the geometric locus of all the points in the interior of an angle equally distanced from the sides of the angle.



Proof. Consider the points $M \in AB$ and $N \in AC$ such that $DM \perp AB$ and $DN \perp AC$.

Part I: Assume (A) and prove (B).

$$\left.\begin{array}{r}\text{By assumption,} \widehat{MAD} = \widehat{DAN}. \\ |AD| = |AD| \\ \widehat{MDA} = 90^\circ - \widehat{MAD} = 90^\circ - \widehat{DAN} = \widehat{NDA}\end{array}\right\} \quad (A.S.A.) \Rightarrow \triangle MAD \equiv \triangle DAN$$

$$\Rightarrow |MD| = |ND|.$$

Thus the distances $|MD|$ and $|ND|$ from D to AB and AC, respectively, are equal.

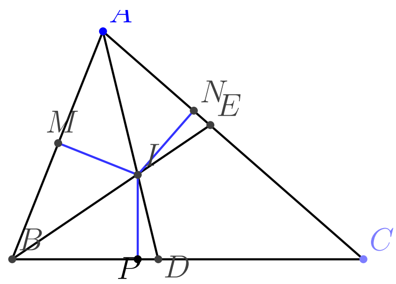
Part II: Assume (B) and prove (A).

$$\left.\begin{array}{r}\text{By assumption,}|MD| = |ND|.\\ |AD| = |AD|\\ \text{Pythagora's Theorem,}|AM| = \sqrt{|AD|^2 - |MD|^2} = \sqrt{|AD|^2 - |ND|^2} = |AN|.\end{array}\right\}$$
$$(S.S.S.) \Rightarrow \triangle MAD \equiv \triangle DAN$$
$$\Rightarrow \widehat{MAD} = \widehat{DAN}.$$

Thus AD is the angle bisector of $\widehat{MAN} = \widehat{BAC}$. $\square$

This universal property of a bisector is helpful in proving an important property of a triangle:

Theorem 3.5. ***The incentre of a triangle*** *All the angle bisectors of the interior angles in a triangle ABC intersect at a point I. This point is called the incentre of the triangle.*



Proof. We first notice that the angle bisectors AD and BE of $\widehat{BAC}$ and $\widehat{ABC}$ respectively, must intersect at a point. Indeed, if they were parallel, then by the theorem on parallel lines, $\widehat{DAB} + \widehat{EBA} = 180°$. But $\widehat{DAB} = \frac{1}{2}\widehat{BAC}$ and $\widehat{EBA} = \frac{1}{2}\widehat{ABC}$, and so we would have $\widehat{BAC} + \widehat{ABC} = 360°$, which is absurd: they are interior angles in triangle ABC and so their sum should be $< 180°$.

We will denote the point of intersection of AD and BE by I. It remains to show that CI is the angle bisector of $\widehat{ACB}$. Let $|IM|$, $|IN|$, and $|IP|$ be the distances from I to AB, AC and BC respectively, with $M \in AB$, $N \in AC$, $P \in BC$.

Since AD is the angle bisector of $\widehat{BAC}$, by the universal property of angle bisectors we have $|IM| = |IN|$.

Since BE is the angle bisector of $\widehat{ABC}$, by the universal property of angle bisectors we have $|IM| = |IP|$

Thus $|IN| = |IP|$ and so, by the universal property of angle bisectors, CI is the angle bisector of $\widehat{ACB}$. $\square$

Definition 3.6. In the proof above, we've seen that $|IM| = |IN| = |IP|$, and so, there exists a circle with centre at I and radius IM. This is called the incircle of $\triangle ABC$. Because $IM \perp AB$, $IN \perp AC$ and $IP \perp BC$, the incircle intersects the sides of $\triangle ABC$ at the points M, N, P only. We say that the incircle is tangent to the sides of $\triangle ABC$. (Indeed, if the incircle intersected AB at two points M and M', then $\triangle IMM'$ would be isosceles with the angles at M and M' both equal to $90°$, which is impossible!).

3.2. The perpendicular bisector.

Definition 3.7. The perpendicular bisector of a segment $[BC]$ is the line perpendicular on $[BC]$ and passing through its midpoint.

MO such that M is the midpoint of BC and $MO \perp BC$.

Construction of the perpendicular bisector. Draw two circles centered at B and C respectively, and of the same radius, which intersect at two points S and T. Then the line ST is the perpendicular bisector of $[BC]$. Indeed, By construction, $|BS| = |CS| = |BT| = |CT|$ and so $BSCT$ is a rhombus. By the theorem on rhombus, we know that $ST \perp BC$ and that ST and BC intersect at their midpoints.

There is an alternative way of characterizing the perpendicular bisector.

Theorem 3.8. ***The perpendicular bisector as a geometric locus*** *Consider a segment $[BC]$ and a point S not on it. The following two statements are equivalent:*
(A) S lies on the perpendicular bisector of $[BC]$.
(B) $|SB| = |SC|$.

In other words, the perpendicular bisector of a segment is the geometric locus of all the points equally distanced from the vertices (endpoints) of the segment.

Proof. Consider the point $M \in BC$ such that $SM \perp BC$.

Part I: Assume (A) and prove (B). By assumption, M is the midpoint of $[BC]$ and $SM \perp BC$. thus

$$\left.\begin{array}{r} \text{By assumption,} |BM| = |CM| \\ \text{By assumption,} \widehat{SMB} = \widehat{SMC} = 90^\circ. \\ |MS| = |MS| \end{array}\right\} \quad (S.A.S.) \Rightarrow \triangle MSB \equiv \triangle MSC$$

$$\Rightarrow |SB| = |SC|.$$

Part II: Assume (B) and prove (A).

$$\left.\begin{array}{r} |SM| = |SM| \\ \text{By construction,} \widehat{SMB} = \widehat{SMC} = 90^\circ. \end{array}\right\}$$

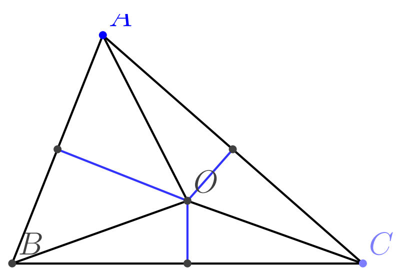
$$\Rightarrow \text{By Pythagora's Theorem,} |BM| = \sqrt{|SB|^2 - |MS|^2} = \sqrt{|SC|^2 - |MS|^2} = |CM|.$$

Thus SM is the perpendicular bisector of $[BC]$. $\square$

This universal property of a perpendicular bisector is helpful in proving an important property of a triangle:

Theorem 3.9. ***The circumcentre of a triangle.*** *All the perpendicular bisectors of the sides in a triangle ABC intersect at a point O.*

This point is called the ***circumcentre*** *of the triangle, because there exists a circle, called the* ***circumcircle*** *of the triangle, of centre O and containing the vertices of the triangle ABC.*



Proof. We first notice that the perpendicular bisectors of $[BC]$ and $[AB]$ must intersect at a point, which we will call O. Indeed, if they were parallel, then by drawing a common parallel to both through B and applying the theorem on parallel lines, we would get $\widehat{ABC} = 180°$, which is absurd.

It remains to show that O lies on the perpendicular bisector of $[AC]$. Since O lies on the perpendicular bisector of $[BC]$, by the universal property of perpendicular bisectors we have $|OB| = |OC|$.

Since O lies on the perpendicular bisector of $[BA]$, by the universal property of perpendicular bisectors we have $|OB| = |OA|$.

Thus $|OA| = |OC|$ and so, by the universal property of perpendicular bisectors, O lies on the perpendicular bisector of $[AC]$.

Since $|OA| = |OB| = |OC|$, the point O is the centre of a circle which contains all vertices of the triangle ABC. □

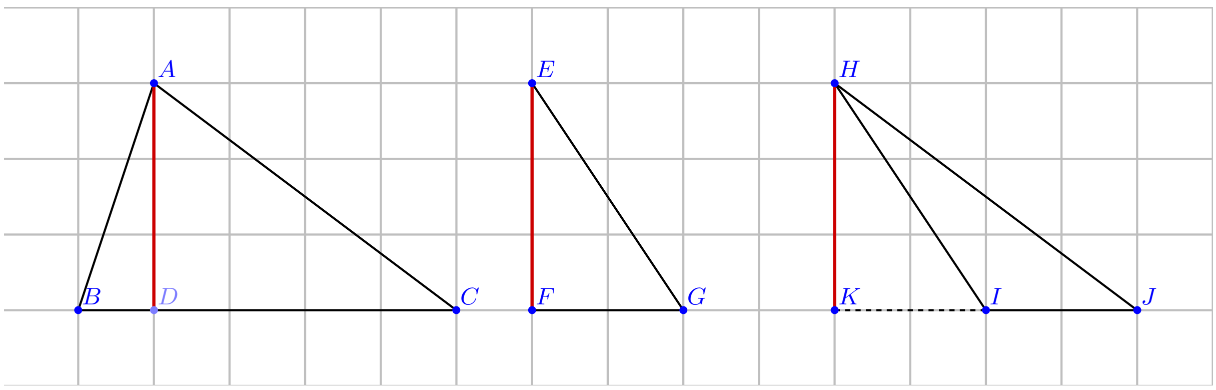
Example 3.10. The circumcentre of a right angled triangle is always the midpoint of the hypothenuse.

Indeed, we can complete $\triangle ABC$ with $\measuredangle A = 90°$ to a rectangle $ABA'C$. From the theorem of rectangles, we know that the diagonals AA' and BC are equal and intersect at their midpoint O. Thus O is equally distanced from A, B, C, A'.

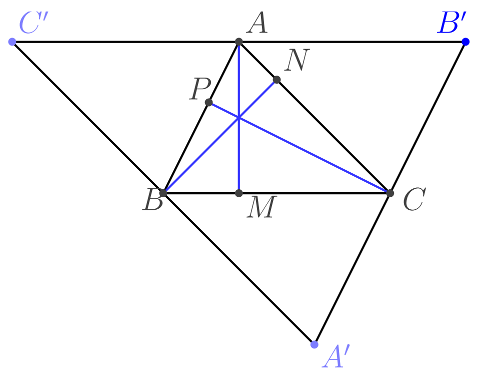
3.3. **Altitudes.**

Definition 3.11. The altitude (height) from the vertex A of a triangle ABC is the line through A perpendicular on BC.

Example 3.12. The lines AD, EF and HK are altitudes in the three distinct triangles below:



Theorem 3.13. *All the altitudes of a triangle intersect at a point H. This point is called the* ***orthocenter*** *of the triangle.*



Proof. Through each of the vertices of the triangle ABC we draw a parallel to the opposite side. By intersecting these lines we get another triangle $A'B'C'$, such that $A \in B'C'$, $B \in A'C'$, $C \in A'B'$.

The proof is based on the observation that the altitudes in the triangle ABC are perpendicular bisectors in triangle $A'B'C'$. Since we have shown that the perpendicular bisectors of a triangle are concurrent, it will follow that the altitudes in the triangle ABC are concurrent at a point H.

Indeed, $ABCB'$ and $ACBC'$ are parallelograms, so by the theorem on parallelograms, $|AB'| = |BC|$ and $|BC| = |AC'|$, which imply $|AB'| = |AC'|$. Moreover, since $BC \parallel B'C'$, it follows that the altitude AM of the triangle ABC is also perpendicular to $[B'C']$, and passing through its midpoint A. Thus AM is the perpendicular bisector of $[B'C']$. Similarly, the altitude BN is the perpendicular bisector of $[A'C']$ and the altitude CP is the perpendicular bisector of $[A'B']$. ☐

As in the case of angle and perpendicular bisectors, it would be nice if we could express an altitude as a geometric locus, i.e. if we could decide whether a point H is on the altitude from A to BC based on measurements involving only H, A, B and C. Given a protractor, we could simply check whether the angle between the lines AH and BC is $90°$. Given only a ruler, we could make use of Pythagora's theorem to check whether the same angle is $90°$. Since our measurement should involve only segments made by H, A, B and C, we will come up with a slightly complicated condition obtained by applying Pythagoras' theorem a number of times and canceling some irrelevant terms.

Lemma 3.14. *Consider $\triangle ABC$. We take a point D on the line BC.*

$$AD \perp BC \Longleftrightarrow |AB|^2 - |AC|^2 = |DB|^2 - |DC|^2$$

Proof. **Part I.** Assume $AD \perp BC$. Prove the relation above. Indeed, $\hat{D} = 90°$. By Pythagoras' theorem in $\triangle ADB$ and $\triangle ADC$ we have:

$$\begin{aligned} |AD|^2 + |DB|^2 &= |AB|^2 \\ |AD|^2 + |DC|^2 &= |AC|^2. \end{aligned}$$

After subtracting the two equations term by term and canceling out $|AD|$:

$$|DB|^2 - |DC|^2 = |AB|^2 - |AC|^2.$$

Part II. Assume $|DB|^2 - |DC|^2 = |AB|^2 - |AC|^2$. We'll prove $AD \perp BC$. Indeed, assume D' is the point on BC such that $AD' \perp BC$. Then by part I,

$$\begin{aligned} |D'B|^2 - |D'C|^2 &= |AB|^2 - |AC|^2, \text{while} \\ |DB|^2 - |DC|^2 &= |AB|^2 - |AC|^2 \end{aligned}$$

by assumption. Hence $|D'B|^2 - |D'C|^2 = |DB|^2 - |DC|^2$, or equivalently,

$$(|D'B| - |D'C|)(|D'B| + |D'C|) = (|DB| - |DC|)(|DB| + |DC|). \tag{3.1}$$

But $|D'B| + |D'C| = |DB| + |DC| = |BC|$ if both D and D' are inside the segment $[BC]$, (which happens when $\triangle ABC$ is acute-angled), or $|D'B| - |D'C| = |DB| - |DC| = \pm|BC|$ if D and D' are outside the segment $[BC]$, both on the same side of the vertices B and C (which is the case when $\hat{B} > 90°$ or $\hat{C} > 90°$).

After cancelling out the equal factors in equation (3.1), we have both

$$\begin{aligned} |D'B| + |D'C| &= |DB| + |DC| \text{ and} \\ |D'B| - |D'C| &= |DB| - |DC|. \end{aligned}$$

which added/subtracted yield $|D'B| = |DB|$ and $|D'C| = |DC|$, hence $D = D'$.

We note that it would be impossible for one of D, D' to be inside the segment $[BC]$ and the other outside. Indeed, if for example D' is outside and D inside, then $|D'B| - |D'C| = |DB| + |DC| = |BC|$. After canceling these factors in equation (3.1), we'd get $|D'B| + |D'C| = |DB| - |DC|$ which is impossible as the positions of D' and D imply $|D'B| + |D'C| \geq |D'B| > |BC| > |DB| > |DB| - |DC|$.

$\square$

Definition 3.15. In the right-angled triangle $\triangle ADB$ with $\hat{D} = 90°$ we define

$$\cos B := \frac{|BD|}{|AB|} \text{ and } \sin B := \frac{|AD|}{|AB|}.$$

Note: $\cos B$ and $\sin B$ thus defined seems to depend on the choice of the triangle $\triangle ABD$. We will prove later that they depend in fact only on the measure of the angle $\hat{B}$.

Corollary 3.16. ***The cos formula.*** *Consider $\triangle ABC$ with the side lengths denoted by $|AB| = c$, $|AC| = b$ and $|BC| = a$. Let D be the point on BC such that $AD \perp BC$. Then*

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

Proof. Consider the case when $\triangle ABC$ is acute angled. From Lemma 3.14 we have $|DB|^2 - |DC|^2 = |AB|^2 - |AC|^2 = c^2 - b^2$. But $|DC| = |BC| - |DB| = a - |DB|$. Substituting this in the equation above we get: $|DB|^2 - (a - |DB|)^2 = c^2 - b^2$.. Solving for $|DB|$ we get $|DB| = \frac{a^2+c^2-b^2}{2a}$. $\square$

Corollary 3.17. ***Heron's formula for the area of a triangle.*** *Consider $\triangle ABC$ with the side lengths denoted by $|AB| = c$, $|AC| = b$ and $|BC| = a$. Then*

$$Area(ABC) = \sqrt{p(p-a)(p-b)(p-c)}$$

where $p = \frac{1}{2}(a + b + c)$ is the semiperimeter of $\triangle ABC$.

Proof. Continuing with the calculations from the previous proof, we apply Pythagoras' theorem in $\triangle ADB$ with $|AB| = c$ and $|DB| = \frac{a^2+b^2-c^2}{2a}$ to get

$$\begin{aligned}
|AD| &= \sqrt{|AB|^2 - |DB|^2} = \sqrt{c^2 - \frac{(a^2 + c^2 - b^2)^2}{4a^2}} = \sqrt{\frac{4a^2c^2 - (a^2 + c^2 - b^2)^2}{4a^2}} \\
&= \frac{\sqrt{(2ac - a^2 - c^2 + b^2)(2ac + a^2 + c^2 - b^2)}}{2a} = \frac{\sqrt{(b^2 - (a-c)^2)((a+c)^2 - b^2)}}{2a} \\
&= \frac{\sqrt{(b + a - c)(b - a + c)(a + c - b)(a + c + b)}}{2a} = \frac{4\sqrt{p(p-a)(p-b)(p-c)}}{2a}
\end{aligned}$$

As $|AD|$ is the height in $\triangle ABC$ with basis $|BC| = a$, we get the formula for the area as above.

$\square$

Theorem 3.18. ***The altitude as a geometric locus*** *Let $\triangle ABC$ be a triangle and H a point in the plane. Then H is on the altitude from A to BC $\Longleftrightarrow$*

$$AH \perp BC \Longleftrightarrow |AB|^2 - |AC|^2 = |HB|^2 - |HC|^2$$

Proof. Let D be the intersection of the lines AH and BC.

Part I Assume $AH \perp BC$. We will prove the formula above. Indeed, applying Lemma 3.14 to $AD \perp BC$ and then $HD \perp BC$ we get

$$|AB|^2 - |AC|^2 = |DB|^2 - |DC|^2 = |HB|^2 - |HC|^2$$

Part II Assume $|AB|^2 - |AC|^2 = |HB|^2 - |HC|^2$. We will prove $AH \perp BC$. Let D, D' be points on BC such that $AD \perp BC$ and then $HD' \perp BC$. Thus by Lemma 3.14,

$$|AB|^2 - |AC|^2 = |DB|^2 - |DC|^2 \text{ and } |HB|^2 - |HC|^2 = |D'B|^2 - |D'C|^2.$$

We assumed $|AB|^2 - |AC|^2 = |HB|^2 - |HC|^2$, and so $|DB|^2 - |DC|^2 = |D'B|^2 - |D'C|^2$ which as before implies $D = D'$. Thus the lines $AD \perp BC$ and $HD' \perp BC$ have a common point $D = D'$. But only one perpendicular to BC can be constructed from the point D and hence A, D and H must be collinear, $AH \perp BC$.

□

Alternate proof for the Orthocenter Theorem: We prove that the three altitudes in a triangle intersect at a point as follows: First we note that two altitudes BE and CF must intersect at a point. Indeed, assume $BE \parallel CF$. Then $BE \perp AC$ would imply $CF \perp AC$ whereas from definition $CF \perp AB$. But then the triangle formed by the lines AB, AC and CF would have two right angles, which is impossible.

We denote by H the intersection of the two altitudes BE and CF. We will prove that H is also on a point on the altitude from A. Indeed, we can apply the Altitude as geometric locus Theorem:

$$\begin{aligned} BH \perp AC &\iff |BA|^2 - |BC|^2 = |HA|^2 - |HC|^2 \\ CH \perp AB &\iff |CB|^2 - |CA|^2 = |HB|^2 - |HA|^2. \text{ Subtracting,} \\ &\iff |BA|^2 - |CA|^2 = |HB|^2 - |HC|^2, \end{aligned}$$

which implies $AH \perp BC$.

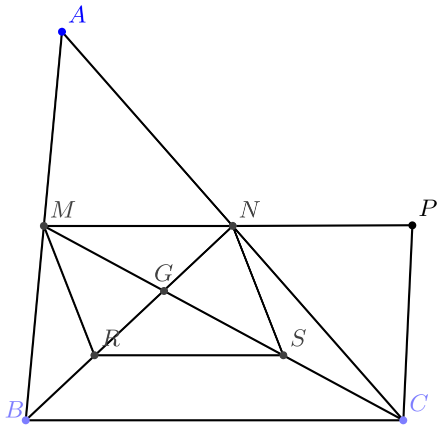
3.4. **Median.**

Definition 3.19. The median from the vertex A of a triangle ABC is the line joining A with the midpoint of $[BC]$.

Definition 3.20. Let ABC be a triangle and consider the midpoint M of the segment AB and the midpoint N of AC. Then MN is called a midline of the triangle ABC.

Proposition 3.21. ***Midlines.*** *Let ABC be a triangle and consider the midpoint M of the segment AB and the midpoint N of AC.*
a) $MN \| BC$ and $|MN| = 1/2|BC|$.
b) Let G be the point of intersection of BN and CM. Then $|BG| = 2|GN| = \frac{2}{3}|BN|$ and $|CG| = 2|GM| = \frac{2}{3}|CM|$.

Proof. a) Extend MN by MP of equal length. By the Theorem on parallelograms (D), $AMCP$ is a parallelogram. This implies that $CP \| AB$ and $|CP| = |AM| = |MB|$. Thus by the Theorem on parallelograms (B), $BMPC$ is a parallelogram too, which implies $MP \| BC$ and $2|MN| = |MP| = |BC|$. This proves a).



b) Draw the midpoints R and S of the segments BG and CG, respectively. Then NS is a midline in $\triangle CGA$ and so by a),

$$NS \| AG \text{ and } |NS| = \frac{1}{2}|AG|.$$

Similarly, MR is a midline in $\triangle BGA$ and so by a),

$$MR \| AG \text{ and } |MR| = \frac{1}{2}|AG|.$$

From the previous two observations, $NS \| MR$ and $|NS| = |MR|$. By the Theorem on parallelograms (B), $MNSR$ is a parallelogram, and so the diagonals intersect each other at midpoints. Thus $|MG| = |GS| = |SC| = 1/2|CG|$ (because S was constructed as the midpoint of the segment CG), and similarly $|NG| = |GR| = |RG| = 1/2|BG|$. □

Theorem 3.22. ***The centroid of a triangle.*** *All the medians of a triangle intersect at a point G. This point is called the* ***centroid*** *of the triangle.*

Proof. Let Q be the midpoint of $[BC]$. With the notations from the Proposition on Midlines, AQ, BN and CM are the median in the triangle ABC, and G is the point of intersection of BN and CM. Assume that BN and AQ intersect at another point G'. We apply the Proposition on Midlines, b) in two cases:

- to G as the point of intersection of BN and $CM \Rightarrow |BG| = \frac{2}{3}|BN|$.
- to G' as the point of intersection of BN and $AQ \Rightarrow |BG'| = \frac{2}{3}|BN|$.

From here it follows that $G = G'$, since $|BG| = |BG'|$ and G, G' are both interior points of $[BN]$. Thus AQ, BN and CM all intersect at G. □

Theorem 3.23. ***The median as a geometric locus*** *A point G in the interior of a triangle $\triangle ABC$ is on one of its medians AM if it forms with the sides AB and AC triangles of equal areas:*

$$G \in AM \textit{ median } \Longleftrightarrow \textit{Area}(ABG) = \textit{Area}(ACG).$$

Proof. **Part I** Assume M is the midpoint of segment BC and that G is a point on the median AM. Then

$$\text{Area}(ABM) = \text{Area}(ACM)$$

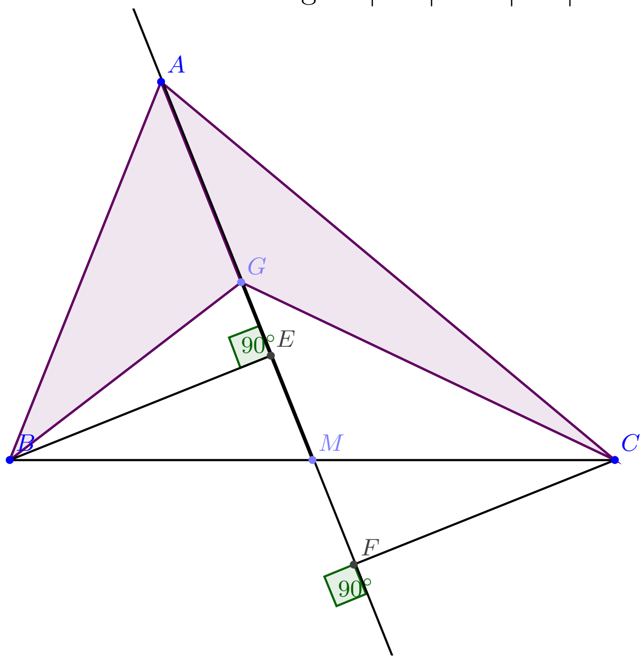
as the triangles have equal bases and the same height. Similarly,

$$\text{Area}(GBM) = \text{Area}(GCM).$$

Subtracting the two equations above yields

$$\text{Area}(ABG) = \text{Area}(ACG).$$

Part II Assume $\text{Area}(ABG) = \text{Area}(ACG)$. As the two triangles have a common side AG, it follows that their heights $|BE|$ and $|CF|$ are equal.



□

Then $\triangle BEM \equiv \triangle CFM$ by AAS, as they have:

- $|BE| = |CF|$;
- $\widehat{BEM} = \widehat{CFM} = 90°$;
- $\widehat{BME} = \widehat{CMF}$ as opposite angles.

Hence, $|BM| = |CM|$ and so AM is median.

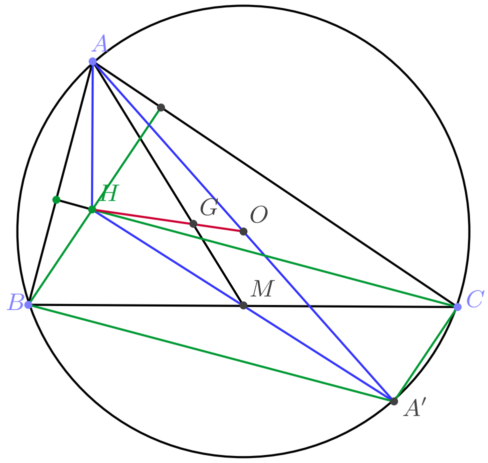
Alternate proof for the theorem of the centroid:

Proof. Two medians BN and CP of $\triangle ABC$ will always intersect at a point G, as they are both in the interior of $\triangle ABC$. We will use the characterization of medians as geometric locus to prove that G is also a point on the median AM. Indeed,

$$\begin{aligned} G \in BN = \text{ median} &\iff \text{Area}(BAG) = \text{Area}(BCG), \\ G \in CP = \text{ median} &\iff \text{Area}(BCG) = \text{Area}(CAG), \end{aligned}$$

Hence $\text{Area}(BAG) = \text{Area}(CAG)$, and so G is also a point on the median AM. □

Theorem 3.24. Euler's line. *The orthocentre H, circumcentre O, and centroid G of any triangle $\triangle ABC$ are collinear and satisfy $|HG| = 2|HO|$.*



Proof.

a) Let M be the midpoint of the segment BC and let AA' be the diameter of the circumcircle of $\triangle ABC$. Then $A'B \perp AB$ and $CH \perp AB$ so $A'B \parallel CH$. As well, and $A'C \perp AC$ and $BH \perp AC$, so $A'C \parallel BH$ and so $BHCA'$ is a parallelogram.

b) Hence M is the midpoint of segment $[HA']$.

c) Hence G is the centroid of $\triangle AHA'$. As such, G is on the median HO and $|HG| = 2|HO|$ by the Theorem of the Centroid.

□

4. Circles

Notation . We will denote by $\mathcal{C}(O, R)$ a circle of center O and radius R.

4.1. **Secants and tangents.**

Definition 4.1. Let $\mathcal{C}$ denote a circle. A **secant** is a line which intersects the circle at two points.

Lemma 4.2. *Let $\mathcal{C}(O, R)$ be a circle of centre O and radius R and let A, B be two points on the circle. Then the perpendicular from O to AB intersects the segment $[AB]$ at its midpoint P, and*

$$|OP|^2 = R^2 - \frac{1}{4}|AB|^2.$$

Proof. Let P denote the point of intersection of AB with the perpendicular from O to AB. Then by applying Pythagora's theorem in the right angled triangles $\triangle OPA$ and $\triangle OPB$ we get

$$|AP|^2 = |OA|^2 - |OP|^2 = |OB|^2 - |OP|^2 = |BP|^2.$$

As $|AP| = \frac{1}{2}|AB|$, the equation above can be rearranged like in the conclusion of the lemma. □

Definition 4.3. A **tangent** is a line which intersects the circle at exactly on point.

We will think of the tangent as the limit of a sequence of secants passing through a fixed point M, and moving gradually further away from the center O of the circle.

Definition 4.4. Let M be a point outside a line d, and let P be a point on d such that $MP \perp d$. We say that P is the **projection** of M on the line d.

Theorem 4.5. *Let M be a point not situated on a circle $\mathcal{C}$ of center O, and let MP be tangent to $\mathcal{C}$, where P is a point on the circle. Then $OP \perp MP$.*

Proof. We consider a secant line AB passing through M. Keeping M as a pivot, we move the line AB towards the outside of the circle. Let P denote the projection of O on the line AB, so that $OP \perp MP$. Then by the previous lemma,

$$|OP|^2 = R^2 - \frac{1}{4}|AB|^2.$$

The line through M, A, B and P is in a tangent position to the circle precisely when $|AB| = 0$, and so $|OP| = R$. In other words, P is on the circle and $OP \perp MP$. □

Conversely, we have:

Theorem 4.6. *Let M be a point not situated on a circle $\mathcal{C}$ with center O, and let P is a point on the circle. Assume $OP \perp MP$. Then MP is tangent to the circle $\mathcal{C}$.*

Proof. We want to show that MP is tangent to the circle $\mathcal{C}$, i.e., by definition, that P is the unique point of intersection of MP with $\mathcal{C}$. Uniqueness is most often proven by contradiction, so we will assume that there exists another point of intersection P'. Then since P and P' are both on the circle, the triangle OPP' is isosceles and as such, $\hat{P'} = \hat{P} = 90°$ (since $OP \perp MP$). But the angles in $\triangle OPP'$ can't sum up to more than $180°$ – thus our assumption of the existence of P' must have been wrong. □

Lemma 4.7. *Let M be a point not situated on a circle $\mathcal{C}$, and let MP and MP' be tangents to $\mathcal{C}$, where P and P' are points on the circle. Then*

$$|MP| = |MP'|.$$

Proof. This follows by applying Pythagora's theorem in triangles OMP and OMP'. □

Lemma 4.8. *Consider two circles $\mathcal{C}_1(O_1, R_1)$ and $\mathcal{C}_2(O_2, R_2)$ with common tangents MP and $M'P'$, where M, M' are points on $\mathcal{C}_1$ and P, P' are points on $\mathcal{C}_2$. Then*

$$|MP| = |M'P'| = \sqrt{|O_1O_2|^2 - (R_1 - R_2)^2}.$$

Hint: Draw $O_1A \perp O_2P$ and $O_1A \perp O_2P'$ and apply Pythagoras' theorem in triangles O_1O_2A and O_1O_2A'.

Theorem 4.9. *Let $C_1(O_1, R_1)$ and $C_2(O_2, R_2)$ be two circles intersecting at two points A and B. Then $O_1O_2 \perp AB$ and O_1O_2 passes through the midpoint of the segment $[AB]$.*

Proof. $\triangle AO_1O_2 \equiv \triangle BO_1O_2$ (case SSS) $\Longrightarrow \widehat{AO_2O_1} = \widehat{BO_2O_1} \Longrightarrow O_1O_2$ is angle bisector in the isosceles triangle O_2AB with $|O_2A| = |O_2B| \Longrightarrow O_1O_2 \perp AB$ and O_1O_2 passes through the midpoint of the segment $[AB]$ (as O_1O_2 must also be perpendicular bisector in triangle O_2AB). □

Theorem 4.10. *Two circles $C_1(O_1, R_1)$ and $C_2(O_2, R_2)$ are called tangent to each other if they intersect at only one point.*

Just like with a circle and a line, we consider tangency of two circles as the limit position of a sequence of secant circles. Hence the following theorem.

Theorem 4.11. *Let $C_1(O_1, R_1)$ and $C_2(O_2, R_2)$ be two circles tangent to each other at the point P. Then O_1, O_2 and P are collinear.*

Proof. Consider a sequence of circles $C_n(O_n, R_1)$ intersecting the circle $C_2(O_2, R_2)$ at two points A_n and B_n, and such that $\lim_{n\to\infty} O_n = O_1$. By the previous theorem, the line O_1O_2 passes through the midpoint P_n of the segment $[A_nB_n]$. On the other hand, $\lim_{n\to\infty} O_n = O_1$ implies $\lim_{n\to\infty} A_n = P$ and $\lim_{n\to\infty} B_n = P$, and so the same is true for midpoints: $\lim_{n\to\infty} P_n = P$ (exercise in coordinate calculations, using the fact that all circles $C_n(O_n, R_1)$ have radius R_1). As $P_n \in O_1O_2$ for all n, it follows that $P \in O_1O_2$.

□

4.2. **Arcs and angles.**

Definition 4.12. An arc $\overset{\frown}{\mathrm{AB}}$ is a portion of a circle bounded by two points A and B on the circle.

Since two points on the circle bound two arcs, the notation is rather imprecise. However, we'll employ it when the choice of arc is evident from the context.

The intuitive notion of arc length is very old, as the ideas of measuring arc length by using a thread, or by rotating a circle along a ruler.

Definition 4.13. Consider a circle $\mathcal{C}(O, R)$. A radian is the measure of an angle $\widehat{AOB}$ such that the arc $\overset{\frown}{\mathrm{AB}}$ which subtends it has length R.

Note: The above definition is written in terms of a chosen circle $\mathcal{C}(O, R)$. In fact, the notion of radian is independent on the choice of a circle, but we will only be able to prove this later using similar triangles.

Notation . For simplicity, we will identify the measure of an arc $\overset{\frown}{\mathrm{AB}}$ with that of the angle $\widehat{AOB}$, and write simply $\overset{\frown}{\mathrm{AB}} = \widehat{AOB}$. To justify this, we note that measuring the length of $\overset{\frown}{\mathrm{AB}}$ as a multiple of the radius will yield the same number as measuring $\widehat{AOB}$ in radians.

Lemma 4.14. ***(Angle on a circle.)*** *Let A, B, C be three points on a circle. Let $\overset{\frown}{BC}$ denote the arc which does not contain A. Then*

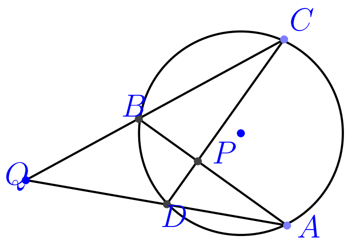
$$\widehat{BAC} = \frac{\widehat{BOC}}{2} = \frac{\overset{\frown}{BC}}{2}.$$

Proof. Let D be the point on the circle such that A, O and D are collinear. Then

- $\widehat{BOD} = 2\widehat{OAB}$, as exterior angle of the isosceles triangle OAB, and
- $\widehat{COD} = 2\widehat{OAC}$, as exterior angle of the isosceles triangle OAC.

If O is in the interior of the angle $\widehat{BAC}$, we add the equations above term by term, and O is outside of the angle $\widehat{BAC}$, we subtract the equations above term by term. In both cases, we obtain $\widehat{BOB} = 2\widehat{BAC}$. □

Corollary 4.15. ***(Internal and external angles)*** *Let $\mathcal{C}$ be a circle of center O and P a point in the interior of $\mathcal{C}$. Let A, B, C, D be points on $\mathcal{C}$ like in the figure below.*



Let Q be the intersection point of the lines AD and BC. Prove that

$$\widehat{APC} = \frac{1}{2}(\overset{\frown}{AC} + \overset{\frown}{BD}) \text{ and } \widehat{AQC} = \frac{1}{2}(\overset{\frown}{AC} - \overset{\frown}{BD}).$$

Proof. $\widehat{APC}$ is exterior angle for $\triangle PBC \Longrightarrow \widehat{APC} = \widehat{PBC} + \widehat{PCB} = \frac{1}{2}(\overset{\frown}{AC} + \overset{\frown}{BD})$ (angles on the circle). $\widehat{ABC}$ is exterior angle for $\triangle QBA \Longrightarrow \widehat{AQB} = \widehat{ABC} - \widehat{BAD} = \frac{1}{2}(\overset{\frown}{AC} - \overset{\frown}{BD})$. □

Lemma 4.16. *Let A, B, C be three points on a circle. Also, let BD be tangent to the circle, with D on the same side of AB as C. Then*

$$\widehat{BAC} = \widehat{DBC}.$$

The proof is left as an exercise.

Definition 4.17. A quadrilateral $ABCD$ is called cyclic if all its vertices are on a circle.

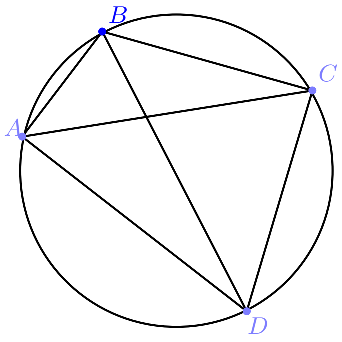
Lemma 4.18. ***(Isosceles trapezoid in a circle.)*** *Let $ABCD$ be a cyclic quadrilateral. The following are equivalent:*
a) $|AD| = |BC|$.
b) $\overset{\frown}{AD} = \overset{\frown}{BC}$.
c) $AB \parallel CD$.

Proof. a) $|AD| = |BC| \Longleftrightarrow \triangle OAD \equiv \triangle OBC$ (case SSS) $\Longleftrightarrow \widehat{AOD} = \widehat{BOC} \Longleftrightarrow$
b) $\overset{\frown}{AD} = \overset{\frown}{BC} \Longleftrightarrow \widehat{ABD} = \frac{\overset{\frown}{AD}}{2} = \frac{\overset{\frown}{BC}}{2} = \widehat{BDC}$ (angles on the circle) $\Longleftrightarrow$
c) $AB \parallel CD$. □

Lemma 4.19. ***(Rectangle in a circle.)*** *Let $ABCD$ be a quadrilateral whose vertices are on a circle $\mathcal{C}(O, R)$. The following are equivalent:*
a) $ABCD$ is a rectangle.
b) AC and BD are diameters of the circle (i.e., A, O, D are collinear, and B, O, C are collinear).

Proof: Exercise.

Theorem 4.20. *Let $ABCD$ be a quadrilateral. The following are equivalent:*
a) The quadrilateral $ABCD$ is cyclic.
b) $\widehat{ABD} = \widehat{ACD}$. (The angle formed by a diagonal with a side is equal with that formed by the other diagonal with the opposite side).
d) $\widehat{ABC} + \widehat{ADC} = 180°$ (The sum of two opposite angles is $180°$).

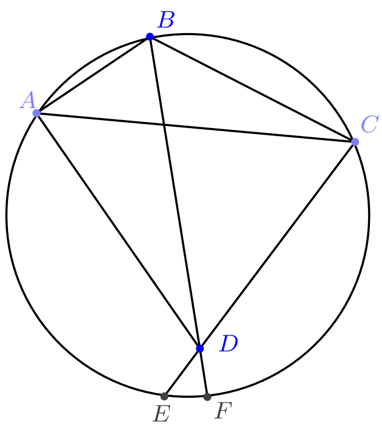


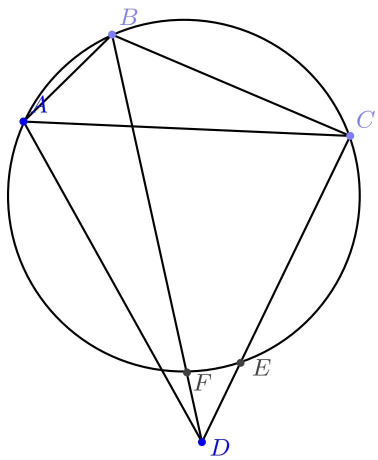
Proof. Part I: we assume a). We prove b) and c). Indeed, on the circle $\mathcal{C}$ containing the vertices A, B, C, D, we have

$$\widehat{ABD} = \widehat{ACD} = \frac{\overset{\frown}{\mathrm{AD}}}{2} \text{ and}$$
$$\widehat{ABC} + \widehat{ADC} = \frac{\overset{\frown}{\mathrm{ADC}} + \overset{\frown}{\mathrm{ABC}}}{2} = \frac{360°}{2} = 180°.$$

Here $\overset{\frown}{\mathrm{ADC}}$ denotes the arc bounded by A and C and containing the point D, while $\overset{\frown}{\mathrm{ABC}}$ denotes the arc bounded by A and C and containing the point B.

Part II: Assume b). Prove a). In this case, we let $\mathcal{C}$ be the circle containing the points A, B, C. This is the circle whose center is the circumcenter O of the triangle ABC (the intersection of the perpendicular bisectors), and whose radius is $|OA|$. We would like to prove that D is also a point on the circle $\mathcal{C}$. Proof by contradiction: assuming D is not on the circle $\mathcal{C}$, let $\mathcal{C}$ intersect the line CD at the point E, the line BD at the point F.





We have

$$\widehat{ABD} = \frac{\overset{\frown}{\mathrm{AF}}}{2} \text{ while } \widehat{ACD} = \frac{\overset{\frown}{\mathrm{AE}}}{2},$$

as angles on the circle. By assumption, we know $\widehat{ABD} = \widehat{ACD}$, and so by the equations above, $\overset{\frown}{\mathrm{AF}} = \overset{\frown}{\mathrm{AE}}$.

However, if D is outside the circle $\mathcal{C}$, then F is inside the arc $\overset{\frown}{\mathrm{AE}}$ and so $\overset{\frown}{\mathrm{AF}} < \overset{\frown}{\mathrm{AE}}$. Contradiction.

However, if D is inside the circle $\mathcal{C}$, then E is inside the arc $\overset{\frown}{\mathrm{AF}}$ and so $\overset{\frown}{\mathrm{AE}} < \overset{\frown}{\mathrm{AF}}$. Contradiction.

Part II: Assume c). Prove a). Similar with the previous part. We let $\mathcal{C}$ be the circle containing the points A, B, C. We would like to prove that D is also a point on the circle $\mathcal{C}$. Proof by contradiction: assuming D is not on the circle $\mathcal{C}$, let $\mathcal{C}$ intersect the line CD at the point E, the line AD at the point L. We have

$$\widehat{ABC} = \frac{\overset{\frown}{\mathrm{ALC}}}{2},$$

as angle on the circle, and

$$\widehat{ACD} = \frac{\overset{\frown}{\mathrm{ABC}} \pm \overset{\frown}{\mathrm{LE}}}{2},$$

as angle which is either internal, or external to the circle. By assumption, we know $\widehat{ABC} + \widehat{ADC} = 180°$, and so by the equations above,

$$\overset{\frown}{\mathrm{ALC}} + \overset{\frown}{\mathrm{ABC}} \pm \overset{\frown}{\mathrm{LE}} = 360°.$$

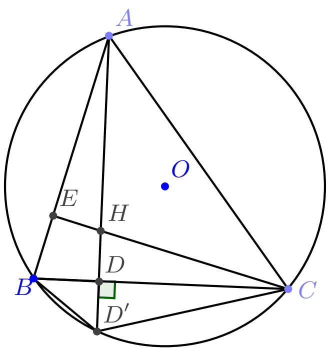
However,

$$\overset{\frown}{\mathrm{ALC}} + \overset{\frown}{\mathrm{ABC}} = 360°$$

as they span the entire circle, so it must be that $\overset{\frown}{\mathrm{LE}} = 0$, meaning that $L = E$. But this would mean that the lines CD and AD intersect the circle at the same point $E = L$. As the intersection of AD and CD is D, we must have $D = E = L$.

□

Example 4.21. Let H be the orthocentre of $\triangle ABC$, and let D' denote the symmetric of H through BC. Then D' is a point on the circumcentre of $\triangle ABC$.



Proof. BC is the perpendicular bisector of $HD' \implies \triangle CDH \equiv \triangle CDD'$. Then $\widehat{AD'C} = \widehat{DHC} = 90° - \widehat{HCD} = \widehat{ABC}$ so the quadrilateral $ABD'C$ is cyclic. (We used $HD \perp BC$ and $CH \perp AB$.) □

5. Similar triangles

Definition 5.1. We say that $\triangle ABC$ and $\triangle A'B'C'$ are similar if their respective angles are equal: $\hat{A} = \hat{A}'$, $\hat{B} = \hat{B}'$, $\hat{C} = \hat{C}'$. we write

$$\triangle ABC \sim \triangle A'B'C'$$

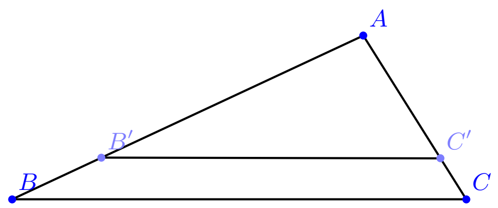
Theorem 5.2. *If* $\triangle ABC \sim \triangle A'B'C'$ *then*

$$\frac{|AB|}{|A'B'|} = \frac{|AC|}{|A'C'|} = \frac{|BC|}{|B'C'|}.$$

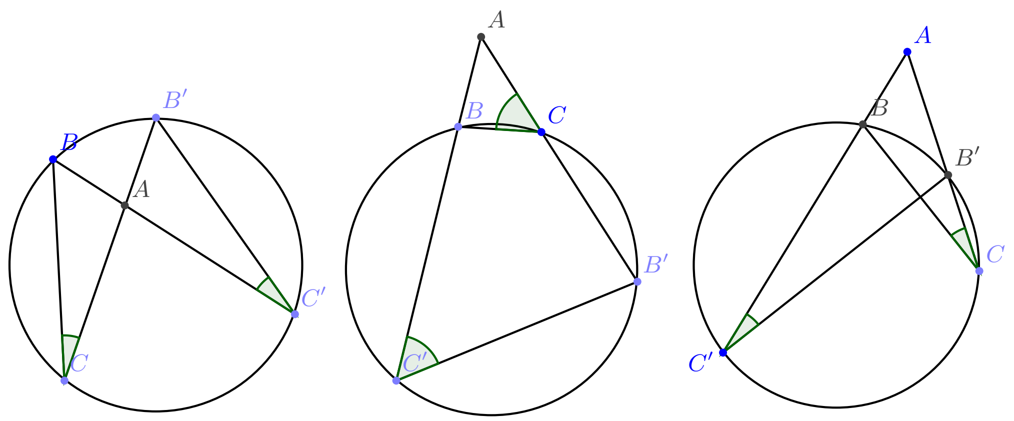
As an application, $\sin\alpha$ and $\cos\alpha$ as defined in trigonometry are independent of the choice of the triangle in which they are computed.

The most common situations when two similar triangles arise are the following:

Theorem 5.3. ***Parallel lines:*** $BC \parallel B'C' \Longrightarrow \triangle ABC \sim \triangle AB'C'$.



Theorem 5.4. ***Anti-parallel lines:***
B, C, C', B' on the same circle and BC' intersects CB' at $A \Longrightarrow \triangle ABC \sim \triangle AB'C'$.



In this case, BC and $B'C'$ are called **anti-parallel**.

Please read the Notes on Areas and solve Ex. Set. 3 for more practice with similar triangles.

In particular, we can solve the following question: Given a circle $\mathcal{C}(O,R)$ and a point P, how can we describe how far the point is from the circle?

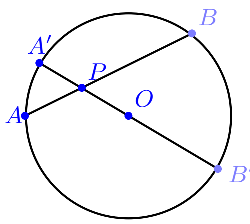
Definition 5.5. The power of a point P with respect to a circle is the product of the segments made by the point P on any chord passing through P.

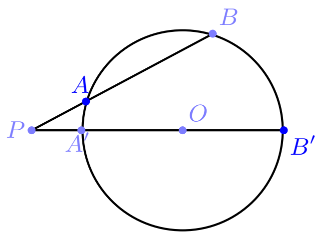
Question (9) from the Ex. Set. 3 can be reformulated as follows:

Theorem 5.6. *The* ***power of a point*** *P with respect to a circle $\mathcal{C}(O,R)$ does not depend on the chord on which it is calculated:*

$$|PA| \cdot |PB| = |PA'| \cdot |PB'| = \pm(|PO|^2 - R^2),$$

$+$ if P is outside the circle and $-$ if P is inside the circle.





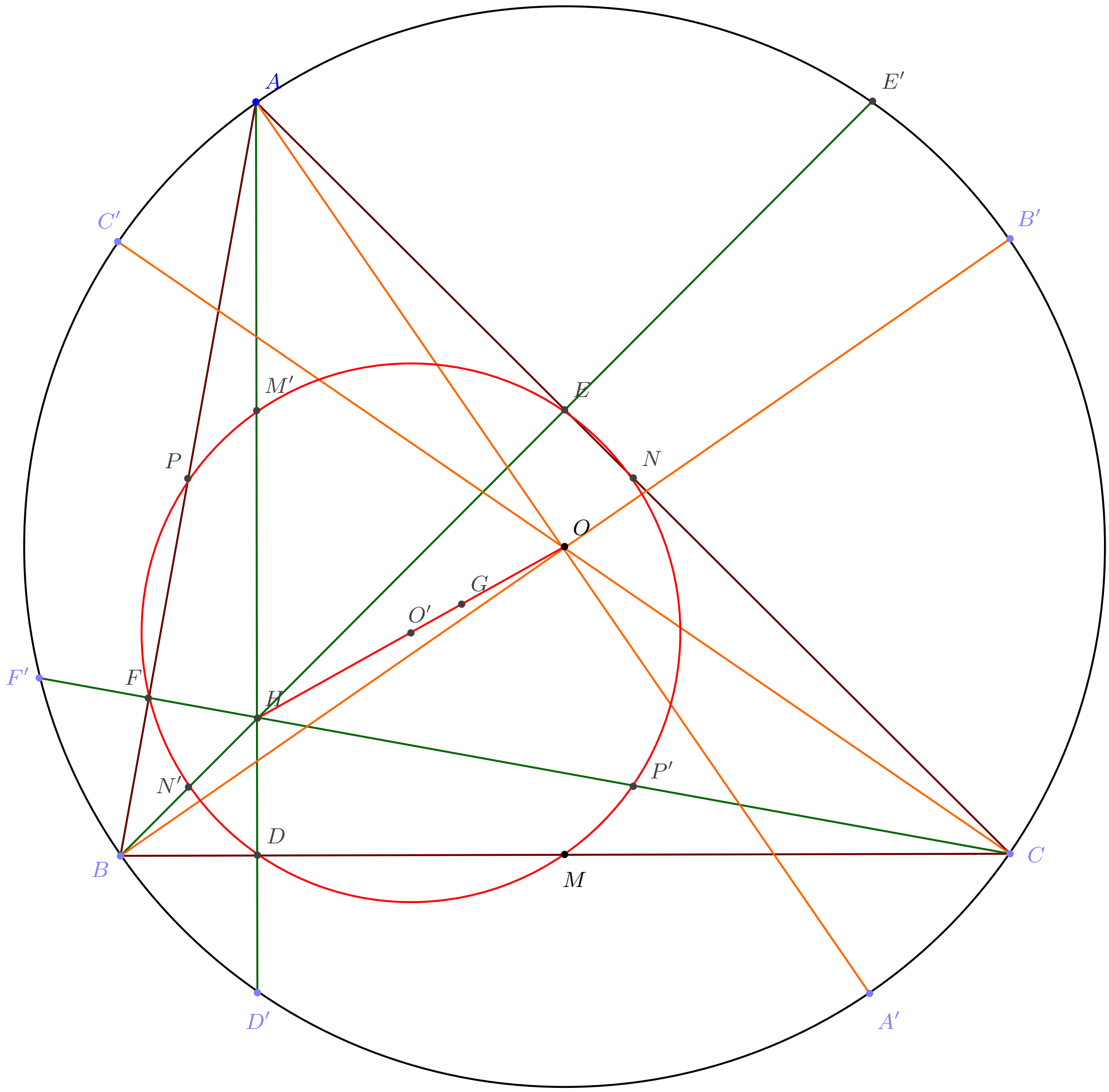
Proof. Due to the equal angles on the circle,

$$\triangle PAA' \sim \triangle PB'B \Longrightarrow \frac{|PA|}{|PB'|} = \frac{|PA'|}{|PB|}$$
$$\Longrightarrow |PA||PB| = |PA'||PB'| = \pm(|OP| - R)(|OP| + R).$$

□

Euler's diagram

One of the nicest structures in the geometry of a triangle is the following. Pieces of it have come up in the Lecture notes, Exercise Sets, Sample Tests etc.



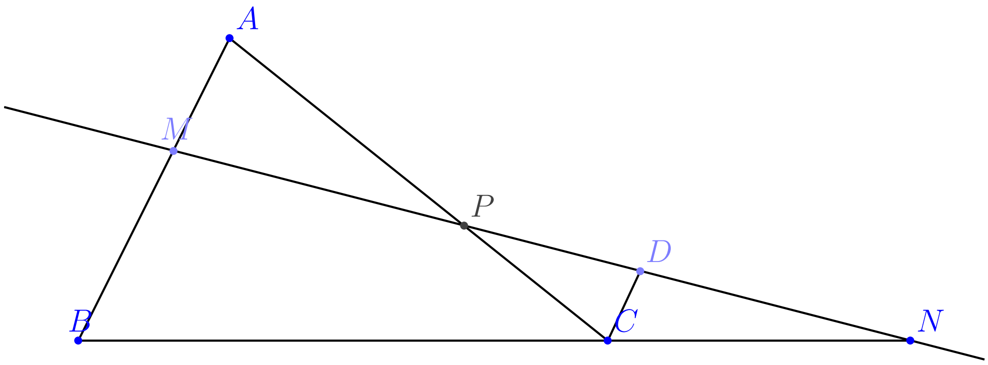
Can you guess the significance of each point? If you were to connect all labeled points, could you find

- at least 19 segments whose midpoints are labeled in?
- at least 9 perpendicular bisectors?
- at least 15 parallelograms which are not rectangles?
- at least 9 isosceles trapezoids?
- at least 6 diameters and 9 rectangles?
- At least 9 pairs of similar triangles sharing H as common vertex? At least 12 pairs of similar triangles sharing A as common vertex?
- At least 8 triangles having G as centroid?

6. Menelaus and Ceva's theorems

Theorem 6.1. *(**Menelaus**) Let M, P be points inside the segments $[AB]$, and $[CA]$, and let N be a point on the line BC, outside of the segment $[BC]$. M, N, P are collinear if and only if*

$$\frac{|AM|}{|MB|}\frac{|BN|}{|NC|}\frac{|CP|}{|PA|} = 1$$



Proof. If the points are collinear, then to prove relation: Let $CD \parallel AB$. Then multiply $\frac{|BN|}{|NC|} = \frac{|MB|}{|CD|}$ with $\frac{|CP|}{|PA|} = \frac{|CD|}{|AM|}$.

If we know relation, assume points are not collinear, let M' be the intersection of lines NP and AB. It has to lie inside the segment $[AB]$ just like M. Then by the previous argument,

$$\frac{|AM'|}{|M'B|}\frac{|BN|}{|NC|}\frac{|CP|}{|PA|} = 1$$

while by assumption

$$\frac{|AM|}{|MB|}\frac{|BN|}{|NC|}\frac{|CP|}{|PA|} = 1$$

hence $\frac{|AM'|}{|M'B|} = \frac{|AM|}{|MB|}$ so $\frac{|AM'|}{|AB|} = \frac{|AM'|}{|AM'|+|M'B|} = \frac{|AM|}{|AM|+|MB|} = \frac{|AM'|}{|AB|}$ so $|AM| = |AM'|$ thus M is M'.

$\square$

Theorem 6.2. *(**Ceva**) Let M, N, P be points inside the segments $[AB]$, $[BC]$ and $[CA]$. The lines AN, BP and CM are concurrent if and only if*

$$\frac{|AM|}{|MB|}\frac{|BN|}{|NC|}\frac{|CP|}{|PA|} = 1.$$

Proof. Part I: We assume the lines AN, BP and CM are concurrent. We apply Menelaus' Theorem twice: once for triangle ABN crossed by line CM and then for triangle ACN crossed by line BP. Multiplying the two ensuing relations yields the formula above.

Part II: We assume the formula above. We let Q be the intersection point of lines BP and CM. Let N' denote the intersection of line AQ with BC. From Part I, we

get:

$$\frac{|AM|}{|MB|}\frac{|BN'|}{|N'C|}\frac{|CP|}{|PA|} = 1.$$

From assumption, we have

$$\frac{|AM|}{|MB|}\frac{|BN|}{|NC|}\frac{|CP|}{|PA|} = 1.$$

Together, these yield

$$\frac{|BN'|}{|N'C|} = \frac{|BN|}{|NC|}.$$

so $P = P'$ since both P and P' are inside the segment $[BC]$. $\square$

References

[1] Silvio Levy, preface to *Flavours of geometry*, MSRI publication, 1997
[2] Euclid's elements, on-line editions:
http://farside.ph.utexas.edu/euclid/Elements.pdf
http://www.gutenberg.org/files/21076/21076-pdf.pdf
[3] Michèle Audin, *Geometry*, Springer Universitext 2003
[4] H.S.M. Coxeter *Introduction to geometry* (2ed., Wiley, 1969)(L)(T)(243s)
[5] Geometry Package GeoGebra available at http://www.geogebra.org/cms/