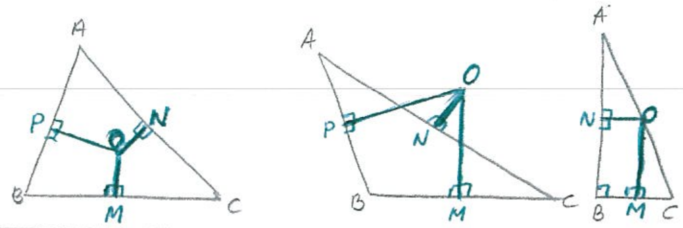
O= Circumcentre of ΔABC

Personal File:

Definition: O = intersection of the perpendicular bisectors:

$|BM| = |MC|$, $|CN| = |NA|$, $|AP| = |PB|$

$OM \perp BC$, $ON \perp AC$, $ON \perp AB$

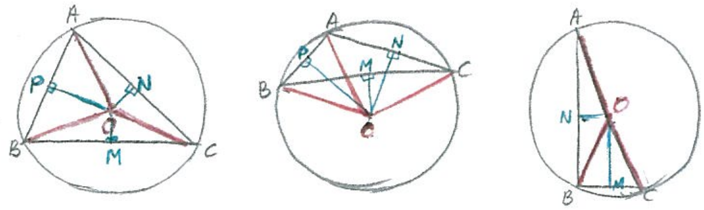


Main Property

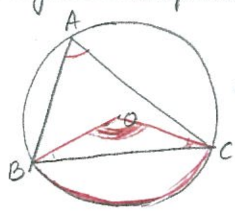
O = centre of circumcircle of ΔABC

$|OA| = |OB| = |OC|$

$\Delta OMB \equiv \Delta OMC$, $\Delta ONC \equiv \Delta ONA$, $\Delta OPA \equiv \Delta OPB$)

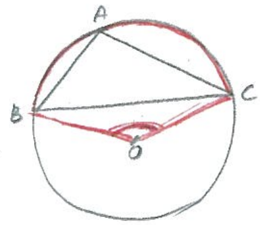


Central Angle Property



$\widehat{BOC} = 2\widehat{BAC} = \widehat{BC}$

$\widehat{OBC} = 90° - \widehat{BAC}$

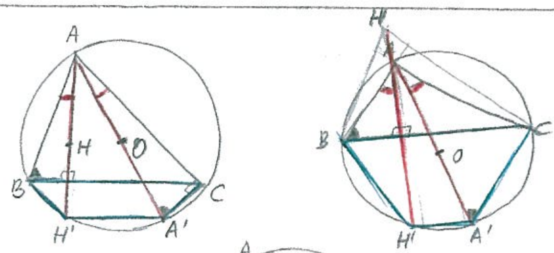


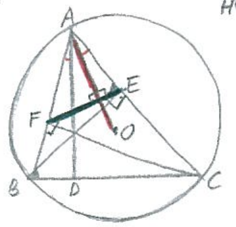
$\widehat{BOC} = 360° - 2\widehat{BAC} = \widehat{BAC}$.

$\widehat{OBC} = \widehat{BAC} - 90°$

Relationship with Orthocentre H

- O and H make equal angles with the sides of ΔABC
- Intersections of AH and AO with circumcircle H' and A' satisfy
  - $H'A' \parallel BC$
  - $H'A'CB$ isosceles trapezium: $|H'B| = |A'C|$
- $OA \perp EF$, where E, F = the feet of altitudes $BE \perp AC$ and $CF \perp AB$

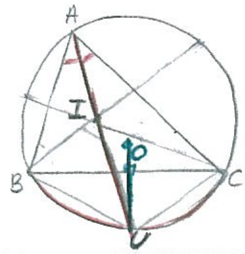




Relationship with Incentre I

The angle bisector from A intersects the perpendicular bisector of BC on the circumcircle of ΔABC

$|UB| = |UC| = |UI|$

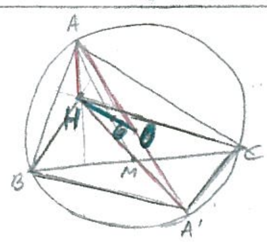


Relationship with Centre of Gravity G

G = centre of gravity of ΔABC

Also G = centre of gravity of $\Delta AHA'$

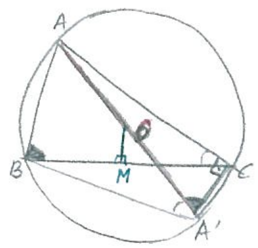
$\Rightarrow$ H, G, O collinear and $|HG| : |GO| = 2:1$



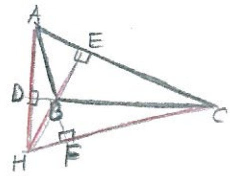
Measurements R = radius of circumcircle

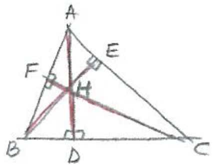
$$\frac{a}{\sin\hat{A}} = \frac{b}{\sin\hat{B}} = \frac{c}{\sin\hat{C}} = 2R = 2|OA| \Rightarrow R = \frac{abc}{4\,Area(ABC)} = \frac{abc}{4\sqrt{p(p-a)(p-b)(p-c)}}$$

$$OM = \sqrt{R^2 - \frac{a^2}{4}}$$



H = Orthocentre of $\triangle ABC$





Definition

H = intersection of altitudes

$AD \perp BC$, $BE \perp AC$, $CF \perp AB$

Cyclic Quadrilaterals

$BCEF$ with circumcentre = midpoint of BC

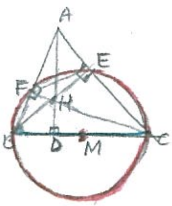
$ACDF$ with circumcentre = midpoint of AC

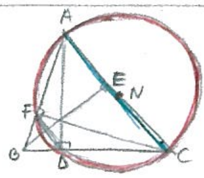
$ABDE$ with circumcentre = midpoint of AB

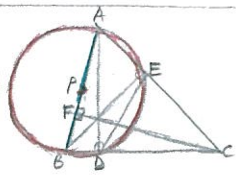
$AEHF$ with circumcentre = midpoint of AH

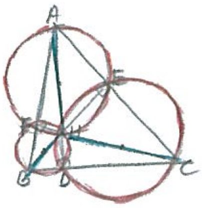
$BDHF$ with circumcentre = midpoint of BH

$CEHD$ with circumcentre = midpoint of CH









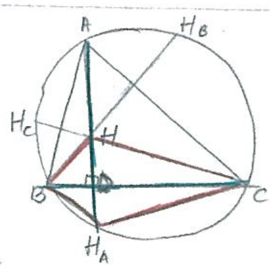
Relationship to Circumcircle

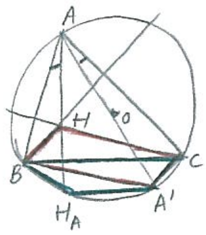
- H_A = symmetric of H with respect to BC is on the circumcircle.

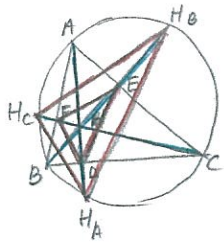
Equivalently, BC = perpendicular bisector of HH_A, where H_A = intersection of AH with circumcircle

$BHCA'$ = paralelogram, where AA' = diameter

- $BCA'H_A$ = isosceles trapezium: $H_AA' \parallel BC$, $|BH_A| = |CA'|$
- H, O make equal angles with the sides: $\widehat{HAB} = \widehat{OAC}$
- H = incentre of $\triangle H_AH_BH_C$
  = incentre of $\triangle DEF$



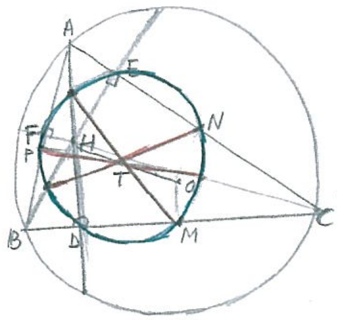




Euler's Circle

- The segments connecting the midpoints of the sides BC, CA, AB with the midpoints of AH, BH, CH are equal and intersect at their common midpoint.
- The points D, E, F where the altitudes intersect the sides, the midpoints of the sides AB, AC, BC and the midpoints of the segments AH, BH, CH are all on the same circle (Euler's).
- The centre of Euler's circle = midpoint of $|HO|$.

Any segment connecting H to a point on the circumcircle of $\triangle ABC$ is cut in half by Euler's circle.



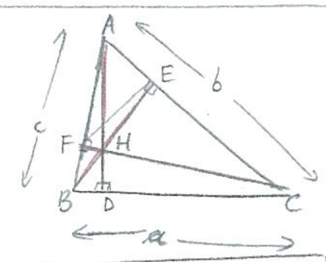
Measurements

$|HB|^2 - |HC|^2 = |DB|^2 - |DC|^2 = |AB|^2 - |AC|^2 = c^2 - b^2$

$$\begin{cases} |BD|^2 - |CD|^2 = c^2 - b^2 \\ |BD| + |CD| = a \end{cases} \Rightarrow \begin{cases} |BD| = \dfrac{a^2+c^2-b^2}{2a} \\ |CD| = \dfrac{a^2+b^2-c^2}{2a} \end{cases} \Rightarrow \boxed{\cos\hat{B} = \dfrac{a^2+c^2-b^2}{2ac}}$$

$$\Rightarrow |AD| = \frac{2}{a}\sqrt{p(p-a)(p-b)(p-c)} \quad \text{where } p = \frac{(a+b+c)}{2}$$

$|EF| = a\cos\hat{A}$ and $|AH| = \dfrac{|AC|\cdot|AE|}{|AD|}$



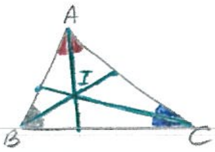
$\boxed{\text{Area}(\triangle ABC) = \sqrt{p(p-a)(p-b)(p-c)}}$

I = Incentre of ΔABC

Definition

I = intersection of angle bisectors

$\widehat{BAI} = \widehat{CAI} = \frac{\widehat{A}}{2}$, $\widehat{ABI} = \widehat{CBI} = \frac{\widehat{B}}{2}$, $\widehat{ACI} = \widehat{BCI} = \frac{\widehat{C}}{2}$

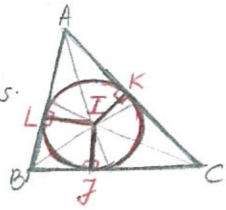


Main Property

I = the point inside ΔABC placed at equal distances from the 3 sides.

$IJ \perp BC$, $IK \perp AC$, $IL \perp AB$ $\Rightarrow |IK| = |IJ| = |IL|$

I = the centre of the incircle. The incircle is inside ΔABC and angent to the 3 sides.



Relationship with Circumcircle

The angle bisector of $\widehat{A}$ and the perpendicular bisector of BC intersect at a point U on the circumcentre of ΔABC.

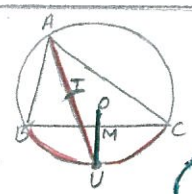
U = midpoint of arc $\widehat{BC}$ = circumcentre of ΔBIC

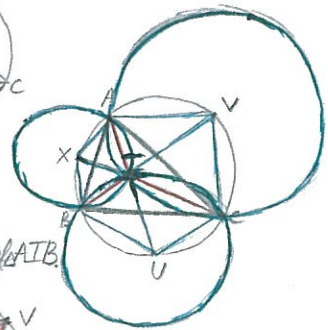
V = midpoint of $\widehat{AC}$ = circumcentre of ΔAIC, X = midpoint of $\widehat{AB}$ = circumcentre of ΔAIB.

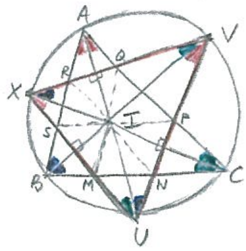
I = orthocentre of ΔUVX

SIMB, PINC, RIQA are rhombuses.

- S, I, P are collinear.
- R, I, N are collinear
- M, I, Q are collinear





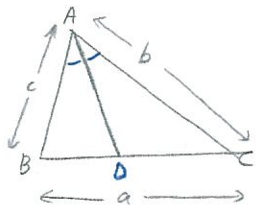


Measurements

AD = angle bisector

$$\Rightarrow \begin{cases} \frac{|BD|}{|DC|} = \frac{|AB|}{|AC|} = \frac{c}{b} \\ |BD| + |DC| = |BC| = a \end{cases} \Rightarrow \begin{cases} \boxed{|BD| = \frac{ac}{b+c}} \\ \boxed{|DC| = \frac{ab}{b+c}} \end{cases}$$

$$\Rightarrow \boxed{|AD| = \frac{2\sqrt{bc \cdot p(p-a)}}{b+c}}$$

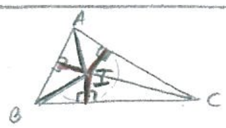


$$\boxed{p = \frac{a+b+c}{2}}$$

Measurements

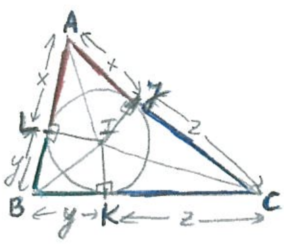
r = radius of incircle

$$\boxed{r = \frac{\text{Area}(\Delta ABC)}{p} = \frac{\sqrt{p(p-a)(p-b)(p-c)}}{p}}$$



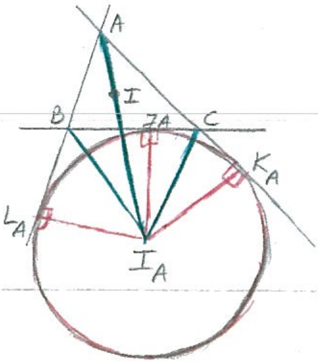
- $|AL| = |AJ| = x$, $|BL| = |BK| = y$, $|CK| = |CJ| = z$

$$\begin{cases} x + y = c \\ x + z = b \\ y + z = a \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2}(b+c-a) = p-a \\ y = \frac{1}{2}(a+c-b) = p-b \\ z = \frac{1}{2}(a+b-c) = p-c \end{cases}$$



I_A, I_B, I_C = Exocentres of ΔABC

Definition



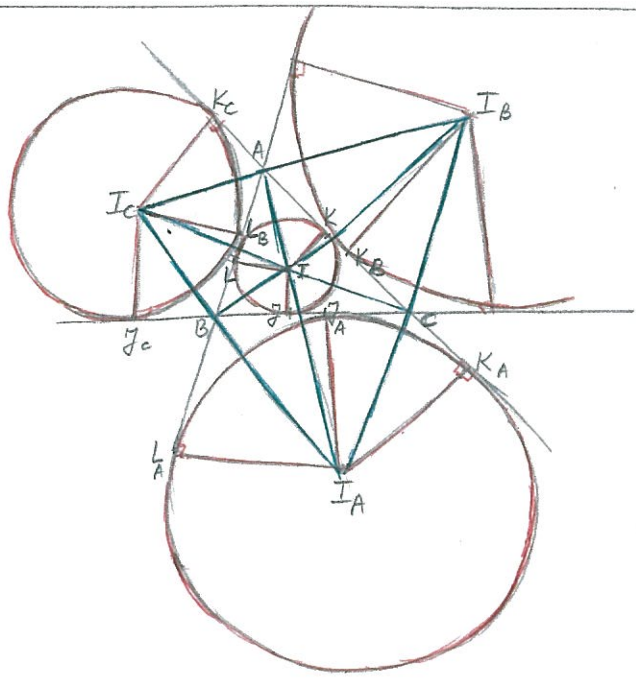
I_A = intersection of the angle bisector of $B\hat{A}C$ with the angle bisectors of the exterior angles of ΔABC at B and C.

Main Property

I_A = the point outside ΔABC but inside $B\hat{A}C$ placed at equal distances from the 3 sides: $|I_AJ_A| = |I_AK_A| = |I_AL_A|$

I_A = the centre of the exocircle at A, which touches the side BC and the lines AB and AC outside the segments $\overline{AB}$ and $\overline{AC}$

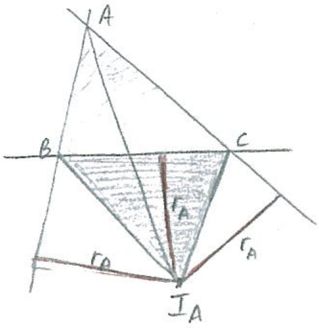
Relationship with Incentre



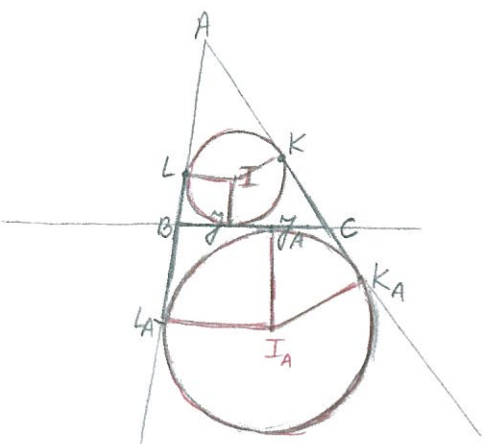
I = orthocentre of $\Delta I_AI_BI_C$

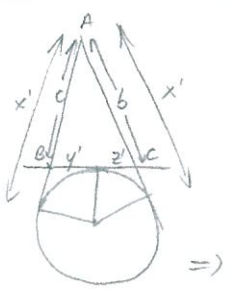
r_A = radius of exocircle centered at I_A

$$\boxed{r_A = \frac{Area(\Delta ABC)}{p-a}}$$



$$\left\{\begin{array}{l} \boxed{|BK| = |BL| = |CJ_A| = |CK_A| = p-b} \\ \boxed{|BJ_A| = |BL_A| = |CJ| = |CK| = p-c} \end{array}\right\} \Rightarrow$$ The common exterior tangents $|LL_A|$ and $|KK_A|$ are equal and also equal to the common internal tangent $|BC|$.





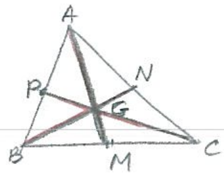
$$\begin{cases} y'+z'=a \\ x'-y'=c \\ x'-z'=b \end{cases} \Rightarrow \begin{cases} y' = \dfrac{a+b-c}{2} = p-c \\ z' = \dfrac{a+c-b}{2} = p-b \\ x' = \dfrac{b+c+a}{2} = p \end{cases}$$

G = Centre of Gravity = Centroid

Personal File

Definition

G = intersection of medians :



$|BM| = |MC| \Rightarrow AM$ median

$|CN| = |NA| \Rightarrow BN$ median

$|AP| = |BP| \Rightarrow CP$ median

Main Property

G divides $\triangle ABC$ into triangles of equal areas:

$$\text{Area}(\triangle ABG) = \text{Area}(\triangle ACG) = \text{Area}(\triangle BCG) = \frac{1}{3}\text{Area}(\triangle ABC)$$

$$\text{Area}(\triangle AGP) = \text{Area}(\triangle BGP) = \text{Area}(\triangle BGM) = \text{Area}(\triangle CGM) = \text{Area}(\triangle CGN) = \text{Area}(\triangle AGN)$$

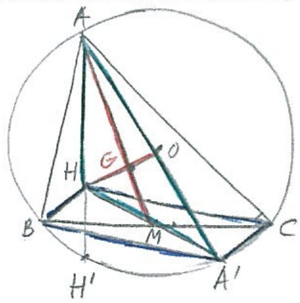
$$= \frac{1}{6}\text{Area}(\triangle ABC)$$

G divides each median into two segments in proportion 2:1

distance from vertex → 2 : 1 ← distance from base:

$$|AG| : |GM| = |BG| : |GN| = |CG| : |GP| = 2:1$$

Relationship with Orthocentre H and Circumcentre O



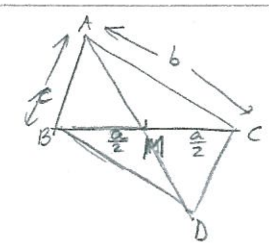
G = centre of gravity of $\triangle AHA'$ where

AA' = diameter in circumcircle

$\Rightarrow$ G is on the line HO and $|HG| : |GO| = 2:1$

Measurements

AM = median



$$\boxed{|AM| < \frac{|AB| + |AC|}{2}}$$

$$\boxed{|AM|^2 = \frac{2b^2 + 2c^2 - a^2}{4}}$$

$$\boxed{|AM| = \frac{a}{2} \Leftrightarrow \hat{A} = 90^\circ}$$

$$|AM| > \frac{a}{2} \Leftrightarrow \hat{A} < 90^\circ$$