Solutions EGMO and UCD Enrichment Programme in Mathematics
Selection Test 4 February 2017

1. Triangle ABC has area S. Denote by M, N and P the midpoints of BC, CA and AB respectively. Prove that

$$2S\left(\frac{1}{AB}+\frac{1}{BC}+\frac{1}{CA}\right) \leq AM+BN+CP < \frac{3}{2}(AB+BC+CA).$$

Solution. We use the fact that in any triangle one side is less than the sum of lengths of other two sides. Applying this fact in triangles ABM and ACM we get

$$AM < AB + BM\,, \quad AM < AC + CM.$$

Adding these two inequalities we find

$$AM < \frac{AB+BC+CA}{2}.$$

Proceeding similarly, we obtain

$$BN < \frac{AB+BC+CA}{2} \quad \text{and} \quad CP < \frac{AB+BC+CA}{2}.$$

Adding the last three inequalities we obtain

$$AM+BN+CP < \frac{3}{2}(AB+BC+CA).$$

Let now D be the feet of the perpendicular from A to BC. Then

$$AM \geq AD = \frac{2S}{BC} \quad \text{and similarly } BN \geq \frac{2S}{CA}\,, CP \geq \frac{2S}{AB}.$$

Adding these three inequalities we obtain

$$AM+BN+CP \geq 2S\left(\frac{1}{AB}+\frac{1}{BC}+\frac{1}{CA}\right).$$

2. A positive integer is said to be *near-square* if it is a product of two positive integers differing by 1. For example, 20 is a near-square because $20 = 4 \times 5$. Prove that every near-square positive integer can be expressed as the ratio of two other near-square positive integers.

Solution.

$$n(n+1) = \frac{(n^2+2n)(n^2+2n+1)}{(n+1)(n+2).}$$

Second solution. (Bernd Kreussler) An alternative solution is

$$n(n+1) = \frac{(n^2-1)\cdot n^2}{(n-1)\cdot n}\,,$$

but this works for $n \geq 2$ only, so we need to mention the solution for the special case $n = 1$, which is $2 = \frac{3\cdot 4}{2\cdot 3}$.

3. Alice and Bob play a game with a string of 2017 pearls. In the first move, Alice cuts the string and Bob chooses a part. Thereafter, the player who chose a part at the end of a move will cut the string in the next move. A player loses if he or she obtains a string with a single pearl such that no more cuts are possible.

Which of the two players has a winning strategy?

Solution.

We claim that for any number of pearls $n \geq 1$ the player whose turn it is to cut when there is an odd number of pearls left is always in a losing position, i.e., the other player can always force a win. Here $n = 2017$, and Alice is in a losing position (i.e., Bob has a winning strategy). We prove our claim by induction on the number of pearls n.

We start with the base case $n = 1$, i.e., a string with one pearl; this is a losing position by definition.

Now, let $n \geq 1$ be odd, and consider a string of n pearls. Assume that for any odd positive integer $k < n$, a string of k pearls represents a losing position. Now, no matter how the first player cuts the string, the cut will always produce an even part. The second player can choose this even part, and cut the string in half, producing two odd parts. The first player is then forced to choose an odd part, which is a losing position by assumption. Therefore, we have proved that the number n is a losing position, and so by the principle of induction the string of n pearls is a losing position for all odd positive integers n.

We conclude that Bob has a winning strategy.

4. The diagonals of the convex quadrilateral $ABCD$ of area 1 intersect at O. If $\frac{BO}{DO} = \frac{1}{2}$ and $\frac{AO}{CO} = \frac{3}{4}$, find the area of triangles AOB, BOC, COD and DOA.

Solution. Denote the area of triangle AOB by x. Then, the areas of triangles DOA, COD, BOC are $2x$, $\frac{8x}{3}$ and $\frac{4}{3}x$ respectively. Since the area of $ABCD$ is 1 we get

$$x + 2x + \frac{8x}{3} + \frac{4x}{3} = 1$$

which yields $x = \frac{1}{7}$. The area of triangles AOB, BOC, COD and DOA is $\frac{1}{7}$, $\frac{4}{21}$, $\frac{8}{21}$ and $\frac{2}{7}$ respectively.

5. Determine with proof all prime numbers p for which $7p + 4$ is the square of an integer.

Solution: The only possibilities are $p = 3$ and $p = 11$. Indeed, if $7p + 4 = n^2$, then $(n - 2)(n + 2) = 7p$. since both 7 and p are prime numbers, it follows that

Case 1. $n - 2 = 1$, $n + 2 = 7p$ with no solution.

Case 2. $n - 2 = 7$, $n + 2 = p$ with $n = 9$, $p = 11$.

Case 3. $n - 2 = p$, $n + 2 = 7$ with $n = 5$, $p = 3$.

Remark: The problem generalizes of course:

Fix q prime and $s < q/2$. Prove that if p is prime and $qp + s^2$ is a perfect square then $p = q \pm 2s$.

Eg., If p is prime and $11p + 9$ is a perfect square then $p = 5$ or 17. etc.

6. (a) Simplify $(x^2 - 1)^2 + (x^2 + 2x)^2 - (x^2 + x + 1)^2$ and then factor the result as far as possible.

(b) Show that there are infinitely many pairs of integers m, n for which $m^2 + n^2 - mn$ is the square of an integer.

Solution:

(a)

$$\begin{aligned}(x^2-1)^2 + (x^2+2x)^2 - (x^2+x+1)^2 &= x^4 - 2x^2 + 1 + x^4 + 4x^3 + 4x^2 \\ &\quad -x^4 - 2x^3 - 3x^2 - 2x - 1 \\ &= x^4 + 2x^3 - x^2 - 2x \\ &= (x^2+2x)(x^2-1) = (x-1)x(x+1)(x+2).\end{aligned}$$

A slightly shorter alternative, which uses the difference of two squares (the first and the third) goes as follows:

$$\begin{aligned}(x^2-1)^2 + (x^2+2x)^2 - (x^2+x+1)^2 &= x^2(x+2)^2 + (-x-2)(2x^2+x) \\ &= x(x+2)\left(x^2+2x-(2x+1)\right) \\ &= x(x+2)(x-1)(x+1)\,.\end{aligned}$$

(b) Let x be any integer. Let $m = x^2 - 1$ and $n = x^2 + 2x$. Then part (a) tells us that

$$m^2 + n^2 - mn = a^2$$

where a is the integer $x^2 + x + 1$. (Since, for example, $x^2 - 1$ and $x^2 + 2x$ are increasing functions of x when $x > 1$ this certainly gives us infinitely many distinct values of m and n.)

7. ABC is an acute triangle. The angle bisector AL, the altitude BH and the median CM are such that $\angle CAL = \angle ABH = \angle ACM$. Find the angles of triangle ABC.

Solution. Assume that L, H and M lie on BC, CA and AB respectively. Denote $\angle CAL = x$. Then $\angle BAL = x$. In triangle ABH we have $\angle AHB = 90^0$, $\angle BAH = 2x$ and $\angle ABH = x$. It follows that $x = 30^0$. Now, in triangle MAC we have $\angle MAC = 60^0$, $\angle MCA = 30^0$ so that $\angle AMC = 90^0$. Thus , the median CM is also an altitude. It follow that triangle ABC is isosceles with $\angle BAC = 60$, so ABC is equilateral. Thus, all its angles are 60^0.

8. The function μ is defined on the set of positive integers as follows:

- $\mu(1) = 1$ and $\mu(p) = -1$ for any prime number p;
- $\mu(ab) = \mu(a)\mu(b)$ for any positive integers with $\gcd(a, b) = 1$;
- $\mu(n) = 0$ if n is a positive integer which is divisible by a square of a prime number.

(For instance $\mu(15) = \mu(3)\mu(5) = 1$ and $\mu(12) = 0$, because 12 is divisible by 2^2).

Prove that for any positive integer $n > 1$, we have $\sum_{d|n} \mu(d) = 0$.

Solution. Denote $M(n) = \sum_{d|n} \mu(d)$.

Let $n = p_1^{a_1} \cdot p_2^{a_2} \cdots p_k^{a_k}$, where $p_1, p_2, \dots, p_k$ are distinct prime numbers and $a_1, \dots, a_k$ are positive integers. Let $d > 0$ be a divisor of n. Then $d = p_1^{b_1} \cdot p_2^{b_2} \cdots p_k^{b_k}$, where $b_1, \dots, b_k$ are nonnegative integers with $0 \le b_i \le a_k$. If $b_i > 1$ for some i, then $\mu(d) = 0$. So, only those d with each $b_i = 0$ or 1 contribute to $M(n)$. For such a d, if $\sum_{i=1}^{k} b_i = r$, then repeatedly using the formula $\mu(ab) = \mu(a)\mu(b)$ if $\gcd(a, b) = 1$, we get $\mu(d) = (-1)^r$.

If $r = 0$ then $d = 1$ and $\mu(d) = \mu(1) = 1$.

If $1 \le r \le k$, there are exactly $\binom{k}{r}$ ways to choose a subset of r elements from $\{p_1, p_2, \dots, p_k\}$ and each of these gives a different divisor d of n and the corresponding contribution is $(-1)^r\binom{k}{r}$. Hence

$$M(n) = \sum_{r=0}^{k} (-1)^r \binom{k}{r}.$$

By the binomial theorem we get $M(n) = (1-1)^k = 0$.

Note. (Bernd Kreussler) One could avoid using the Binomial Theorem as follows.

Given $n > 1$, there exists a prime number p that divides n. Let p^k be the highest power of p that divides n so that we can write $n = mp^k$ where m is a positive integer that is co-prime to p and $k \ge 1$.

The key observation now is that the divisors of m are exactly those divisors of n that are not divisible by p. The result then follows from $\mu(dp) = -\mu(d)$ and $\mu(dp^i) = 0$ if $i > 1$.

Here is the full argument in more detail with fancy notation. Let $\mathcal{M}$ be the set of all positive divisors of m and $\mathcal{N}$ the set of all positive divisors of n. If we denote by $\mathcal{M}_i$ the set that is obtained from $\mathcal{M}$ by multiplying each of its elements by p^i, we have

$$\mathcal{N} = \mathcal{M}_0 \cup \mathcal{M}_1 \cup \mathcal{M}_2 \cup \ldots \cup \mathcal{M}_k \qquad \text{and} \qquad \mathcal{M}_0 = \mathcal{M}\,.$$

Because $\mu(ap^i) = 0$ for all $i > 1$, we obtain $\mu(d) = 0$ for all $d \in \mathcal{M}_i$, if $i > 1$. Hence,

$$\sum_{d|n} \mu(d) = \sum_{d \in \mathcal{M}_0 \cup \mathcal{M}_1} \mu(d) = \sum_{d \in \mathcal{M}} \mu(d) + \mu(dp) = \sum_{d \in \mathcal{M}} \mu(d) - \mu(d) = 0\,.$$

9. Let p, q, r be prime numbers with

$$p < q < r < q + p^4 \qquad \text{and} \qquad pq^2 = r^2 + 1.$$

Find, with proof, all possible values for p, q and r.

Solution. Since $r > q > p \ge 2$, it follows that q and r are both odd. So $r^2 + 1$ is even and thus pq^2 is even. This shows that $p = 2$. Let $r = q + s$ for some positive integer s. From $q < r < q + p^4$ we have $1 \le s < p^4 = 16$. Also, from $r^2 + 1 = 2q^2$ we get

$$(q + s)^2 + 1 = 2q^2,$$

so $q^2 + 2qs + s^2 + 1 = 2q^2$ which yields

$$(q - s)^2 = 2s^2 + 1. \qquad (1)$$

Hence

$$(q - s + 1)(q - s - 1) = 2s^2.$$

This shows that $q - s + 1$ and $q - s + 1$ are both even so $2s^2$ is a multiple of 4, so s is even. Also, $2s^2 + 1$ is a perfect square. Trying $s = 2, 4, 6, 8, 10, 12, 14$ only $s = 2$ and $s = 12$ are acceptable.

If $s = 2$, from (1) we find the solution $p = 2$, $q = 5$ and $r = 7$.

If $s = 12$ again from (1) we find $p = 2$, $q = 29$ and $r = 41$.

10. The positive real numbers a, b, c satisfy the double inequality

$$\frac{b^2}{a+b} + \frac{c^2}{b+c} + \frac{a^2}{c+a} \geq \frac{c^2}{a+b} + \frac{a^2}{b+c} + \frac{b^2}{c+a} \geq \frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a}.$$

Prove that $a = b = c$.

Solution. Looking at the first and the last term of our inequality, we observe that they are equal. Indeed,

$$\begin{aligned}\Big(\frac{b^2}{a+b} + \frac{c^2}{b+c} + \frac{a^2}{c+a}\Big) - \Big(\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a}\Big) \\ = \frac{b^2 - a^2}{a+b} + \frac{c^2 - b^2}{b+c} + \frac{a^2 - c^2}{c+a} \\ = (b - a) + (c - b) + (a - c) = 0.\end{aligned}$$

It follows that

$$\frac{b^2}{a+b} + \frac{c^2}{b+c} + \frac{a^2}{c+a} = \frac{c^2}{a+b} + \frac{a^2}{b+c} + \frac{b^2}{c+a}$$

so

$$\frac{b^2 - c^2}{a+b} + \frac{c^2 - a^2}{b+c} + \frac{a^2 - b^2}{c+a} = 0.$$

Direct calculations show that this implies

$$a^2b^2 + b^2c^2 + c^2a^2 - a^4 - b^4 - c^4 = 0$$

which can be rewritten into $(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2 = 0$. This easily yields $a = b = c$.