Numerical simulations of hysteretic discontinuous flow through porous media

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Abstract

A model for predicting the dynamics of the soil-moisture content in a slab of soil is studied. It consists of an ODE with a time derivative of the Preisach operator. The model takes into account rainfall, transpiration and drainage. Rainfall data taken from the lower Feale watershed in Co. Kerry, Ireland was converted into a piecewise-constant function and used as the input on the right hand side of the ODE. The transpiration rate is estimated from the other measurements at the site. The model was verified by comparing its solution to the soil-moisture data obtained at the same site and shows an improvement upon previous simulations which did not include transpiration.

1 Introduction

The soil plays one of the most important roles in the hydrological cycle. It is a three-phase porous medium, where the phases are particles of ground rock or clay, water and a water vapour air combination. The region of the soil that is unsaturated is known as the vadose zone (or simply the unsaturated zone), and it is in this region that the most interesting nonlinear hysteretic behaviour is observed.

As far back as 1930, it was shown experimentally that there is a hysteretic effect in the relationship of the moisture content and the capillary pressure in soils that are not fully saturated. This effect can be quite strong in certain types of soils, and therefore it is desirable to incorporate hysteresis in the dynamical models describing flows of water through the soil. It is believed that the hysteresis of soil-water is rate-independent when considered on the time-scales of water flow (see [2] and bibliography therein). In recent work [3] a model known as the wedge model was used to successfully fit experimental data from the GRIZZLY database [4], which consisted of measurements of moisture content vs. capillary potential.

Here a model for the discontinuous in time flow of water through porous media such as soils is studied. We investigate dynamics of soil moisture content in presence of hysteresis over a large period of time by numerical simulations. In the previous work [1], a simplified model was considered, which included rainfall and drainage terms. Now we extend the model by adding an evaporation term and compare the new results to the previous simulations.

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The model describes water balance in a fully vegetated slab of soil. The hysteresis is included by means of the Preisach operator, which is a mathematical tool ideally suited for rateindependent hysteretic processes. The flow of water in the model is discontinuous due to the instantaneous switching between atmospheric and soil control in a hydrological system [2], which happens when rain suddenly starts or stops. We use the rainfall, temperature, and other measurements from the lower Feale watershed in Co. Kerry, Ireland [8] to drive the model.

2 The FEST Model

The FEST model [2] describes a fully vegetated slab of soil and transpiring plants. The soil matrix has a standard soil-water characteristic and an unsaturated hydraulic conductivity function. All of the water flow into and out of the soil matrix, namely, transpiration, infiltration, drainage to, and capillary rise from a water table below the slab, is driven by the appropriate difference in total potential energy multiplied by the unsaturated hydraulic conductivity. Thus, the water balance obeys a finite-difference version of Darcy's law for unsaturated flow in porous media. These assumptions yield the following non-linear first-order ordinary differential equation with a prescribed initial condition:

$$L\frac{d\theta}{dt} = f(t), \tag{1}$$

$$\theta(0) = \theta_0, \tag{2}$$

$$f(t) = I(t) - E(t) - D(t), \tag{3}$$

where θ is the volumetric moisture content, I is the rate of infiltration of rain water, E the rate of transpiration from the soil slab, D the rate of drainage or capillary rise below the soil slab, and L is the thickness of a uniform slab of vegetated soil. All rates are cubic meters of water per square meter of soil per unit time, in other words, a velocity. The vegetation excludes evaporation of water from the surface of the soil.

The rate of infiltration I per unit area is assumed to proportional to the difference between the matric potential $\psi = 0$ at saturation in the macropores and the matric potential $\psi < 0$ in the soil, that is,

$$I = \min\left(-\frac{\psi}{A}, Q\right), \tag{4}$$

where A is the associated adjustment time, and Q is the rainfall rate, given as a volumetric rate per unit area. When $-\psi/A < Q$, ponding of water on the surface of the slab is said to occur. For modeling purposes we assume that excess rain runs off immediately into a surface drain. This process is a negative feedback loop driving the water content to saturation and the associated matric potential to zero.

The rate of drainage per unit area is assumed to be driven by the difference in total potential between the center of the soil slab and its base. We assume saturation immediately below the slab and a matric potential of zero. Consequently, we have

$$D = \frac{(\psi - 0) + (-L/2 - (-L))}{B} = \frac{\psi + L/2}{B}, \tag{5}$$

where B is a second adjustment time. Equation (5) can be interpreted as a negative feedback loop with the local equilibrium $\psi = -L/2$. At this equilibrium the matric forces in the soil hold a quantity of water against gravity.

The rate of transpiration $E(t)$ has the form

$$E = \frac{ET(t)}{C}, \tag{6}$$

where C is a third adjustment time, and $ET(t)$ is the cumulative evapotranspiration calculated using the standard Penman-Monteith FAO-56 formula. The formula uses the hourly time-series of net-radiation, soil heat flux, rainfall, pressure, wind-speed, air temperature and air vapor pressure, all measured at the same location [8].

The model is closed with the appropriate wedge model, described in the next section. In summary, the system (1)–(3) has the form

$$\dot{\theta} = f(t, \psi(t)), \tag{7}$$

with $\theta(t) = (\mathcal{P}\psi)(t)$, where $\mathcal{P}$ is a Preisach operator (see [5]).

Equations involving the derivative of the output of the Preisach operator are studied in hydrology, economics, and power electronics. Questions of the existence and uniqueness of solutions of (7) are addressed in [5, 6]. In [7], an algorithm that uses high-order integration methods to obtain numerical solutions of (7) is proposed.

3 Wedge model for the Preisach operator

In this section a brief description of the measure density of the Preisach operator used in equation (7) is provided. The wedge model is based on an equation which fits the non-hysteretic part of the hysteresis curves. From the different known approaches to this (see e.g. [4]) the van Genuchten equation was selected:

$$\theta(\psi) = \theta_r + (\theta_s - \theta_r)\left(1 + \left(\frac{\psi}{\psi_g}\right)^n\right)^{-m}, \tag{8}$$

where θ_r is the residual water content, θ_s is the saturated water content, ψ_g is the van Genuchten pressure-head scale parameter, and m and n are two dimensionless water retention shape parameters. For our model $\theta_r = 0$ and $\theta_s = 1$. The number of parameters was reduced with the relationship $m = 1 - 1/n$, which is valid for a range of soil types [4].

The measure density is constructed as a one-parameter family, distributed between the line $\beta = \alpha$ and the line $\beta = \gamma\alpha$, where $0 \leq \gamma \leq 1$ is a parameter of the model. Moreover, the distribution is chosen to be uniform within the fragment $[\psi, \gamma\psi]$ of any vertical lines of constant $\alpha \equiv \psi$.

The parameters m and ψ_g in the Eq. (8) are fitted to the main drying curve (drying from a fully saturated state), and the measure is defined by

$$\rho_\gamma^W(\alpha, \beta) = \frac{\theta'(\alpha)}{\alpha(1-\gamma)}. \tag{9}$$

Eq. (9) can be rewritten as

$$\rho_\gamma^W(\alpha, \beta) = \frac{C}{\gamma - 1} F(\alpha),$$

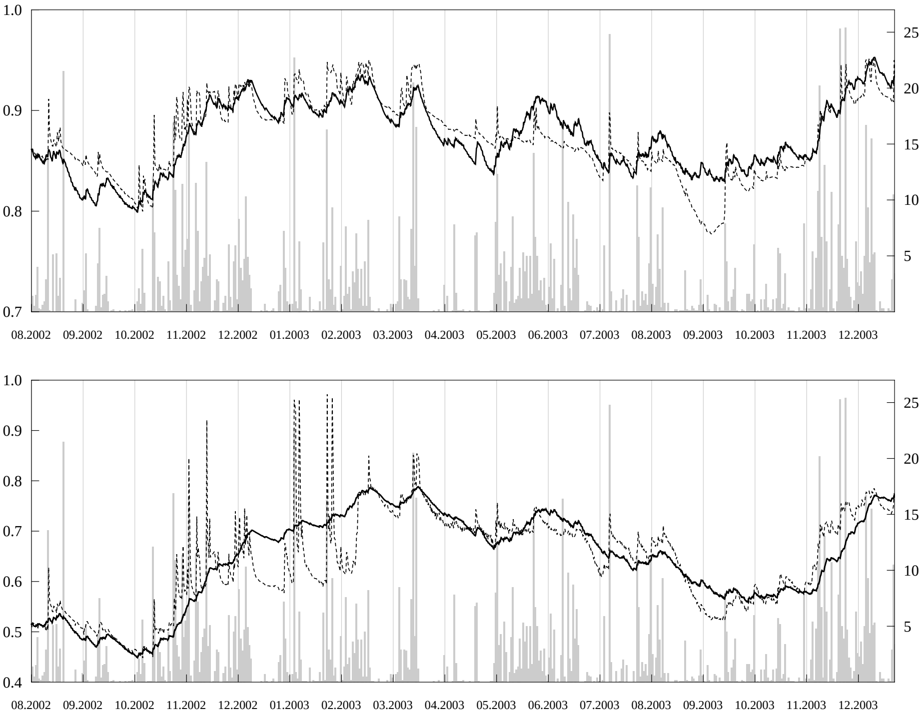

Figure 1: **Comparison with numerical simulation.** Numerical solution of (1)–(3), solid, is plotted against water content data, dashed, for two sets of measurements. In the first plot, $A = 1.2 \times 10^6$ s, $B = 2.6 \times 10^6$ s, $C = 9 \times 10^6$ s, and $L = 0.3$ m. In the second plot, $A = 1 \times 10^6$ s, $B = 1.4 \times 10^7$ s, $C = 4 \times 10^6$ s, and $L = 0.6$ m. For both plots, $n = 5$, $\psi_s = -0.2$ m, and $\gamma = 0.5$ cm/cm. The rainfall data are represented as a bar chart, with each bar corresponding to one day.

where

$$C = \frac{mn\theta_s}{\psi_g^2} \quad \text{and} \quad F(\alpha) = -\left(\frac{\alpha}{\psi_g}\right)^{n-2} \left(1 + \left(\frac{\alpha}{\psi_g}\right)^n\right)^{-1+\frac{1}{n}}.$$

This density is non-zero for $\alpha < \beta < (1-\gamma)\alpha$, where $\alpha < \beta < 0$, and is zero otherwise. The definition (9) implies that the main drying curve as given by Eq. (8) is exactly reproduced by the Preisach operator. The parameter γ is then used to fit the output of the operator to the wetting curves of the soil.

4 Numerical simulations

Figure 1 shows a numerical solution of the system (1)–(3) coupled with the wedge Preisach measure (9). The solution is obtained by means of a C++ program based on the algorithm from [7]. The infiltration $Q(t)$ is constructed as a piecewise-constant function, using rainfall data collected from the lower Feale watershed in County Kerry, Ireland [8]. This data, which covers the period from

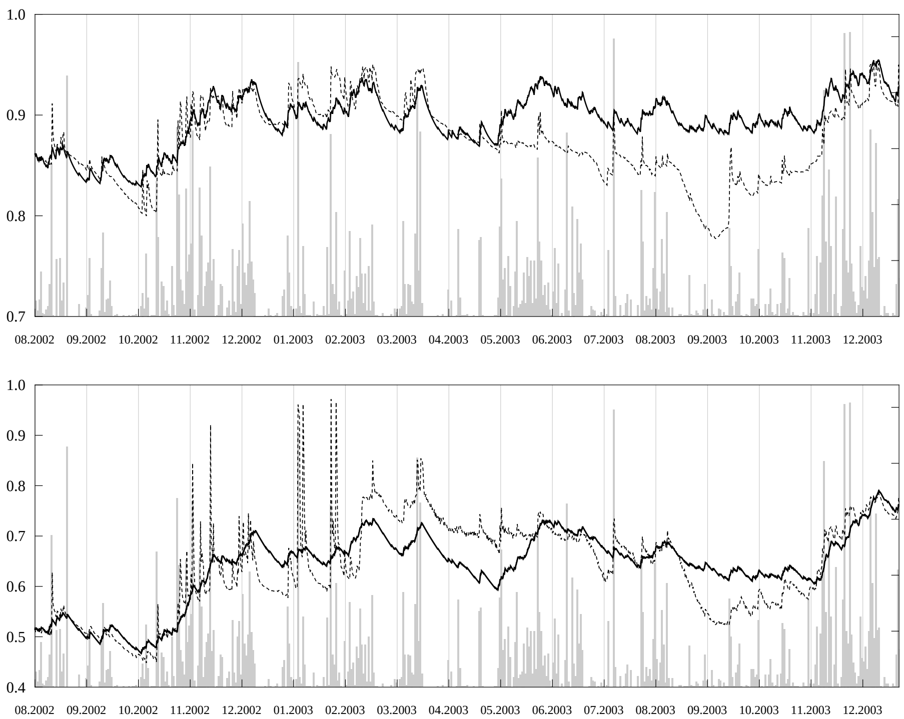

Figure 2: **Results from a previous model.** Numerical solution of a version of (1)–(3) that does not include evaporation, solid, is plotted against water content data, dashed, for two sets of measurements. In the first plot, $A = 9 \times 10^5$ s, $B = 1.5 \times 10^6$ s, and $L = 0.3$ m. In the second plot, $A = 6.5 \times 10^5$ s, $B = 3.7 \times 10^6$ s, and $L = 0.6$ m. For both plots, $n = 5$, $\psi_s = -0.2$ m, and $\gamma = 0.5$ cm/cm. The rainfall data are represented as a bar chart, with each bar corresponding to one day.

August 2002 to February 2004, has a resolution of one hour. The values of $Q(t)$ are proportional to millimeters of rain detected during each time interval. The evapotranspiration $ET(t)$ is also a piecewise-constant function constructed from the output of the FAO-56 formula. The parameters for the Preisach measure are selected to be the typical values for the soil type where the measurements are taken.

The model is verified using the soil-water content data collected at the same location for the same period of time. The two separate sets of water content values are obtained at different depths, with L treated as a parameter of the model. We choose $L = 0.3$ m for the first data set, and $L = 0.6$ m for the second. The remaining parameters, the adjustment times A, B, and C, are selected to optimize the fit between the output of the model and the experimental data.

The results of the simulations show good qualitative agreement with the moisture content measurements, see Figure 1. Note that the measured water content data has visible spikes, which are not reproduced by the model, and which may be artifacts of the measuring system. When compared to the results of simulations with a previous version of the model, see Figure 2, we see that the fit between the model output and the measured moisture content data has been improved, especially in the time period corresponding to summer, where the effects of evaporation are the strongest.

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