Asynchronous systems: a short survey and problems*

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Looking at the dynamics of the system described by the equation

$$x(n+1) = f(x(n))$$

one may say that coordinates of the vector $x = \{x_1, x_2, \ldots, x_N\}$ *are updated* synchronously*. What happens with the system if coordinates of the vector* x *are updated* asynchronously*, i.e., if at a given moment* n *only coordinates with indices* i *from some set* $\omega(n) \subseteq \{1, 2, \ldots, N\}$ *are changed in accordance with the law*

$$x_i(n+1) = f_i(x(n))$$

while others remain intact? This is the main topic which is discussed in the paper.

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1 Preface

The theory of asynchronous systems got its rather distinctive shape about 20 years ago. It was grounded on quite practical problems concerning functioning of distributed computational networks and from the very beginning demonstrated plenty of mathematically difficult, though easily formulated, problems. Examples of systems, for which the problem of synchronization is acute, are complex digital electronic devices, multiprocessor systems, distributed digital networks, discrete-time models of market economy, etc. So, from different points of view it should be very attractive field of investigation for mathematicians. Nevertheless, until now only few of them are familiar with asynchronous systems. To overcome this, on the 6th International Conference on Difference Equations and Applications (August 2001, Augsburg, Germany) there was delivered an 'educational' talk "Asynchronous systems: an intersection point of 'easy questions' with difficult solutions" [15]. An up-to-date version of an analogous survey lecture given by the author during his visit of Boole Centre for Research in Informatics, Cork, Ireland, is presented below.

In the paper, the main attention is paid to discussion of the problem of how asynchronism affects stability of the system. Examples show [1] that all possible combinations of stability/instability for the pair 'synchronous/asynchronous system' may occur. And also, simple examples demonstrate that the problem of investigation of stability for asynchronous system is more complicated than for synchronous one, even in the linear case. Nevertheless, in some situations asynchronous systems possess more robust properties than synchronous ones. Formal explanation of this fact is presented, and various methods of stability investigation for asynchronous system are discussed.

2 Introduction

The simplest, and at the same time the most important object of investigation in the theory of difference equations is the autonomous equation

$$x(n+1) = f(x(n))$$

which is often used to describe the dynamics of a system
where $\{T_n\}$ are discrete moments at which the state $\xi(t)$ of the object is *instantaneously* updated by the controller in accordance to the law

$$\xi(T_n + 0) = f(\xi(T_n - 0)) \qquad (1)$$

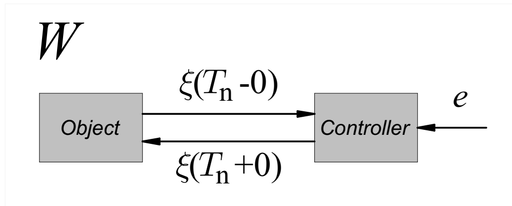

Figure 1: General dynamical system

and the connection between 'physical' state vector $\xi(t)$ and 'abstract' one $x(n)$ is established by the relation $x(n) := \xi(T_n - 0)$.

Clearly, Fig. 1 is very simplified and schematic. A more realistic situation is represented on Fig. 2.

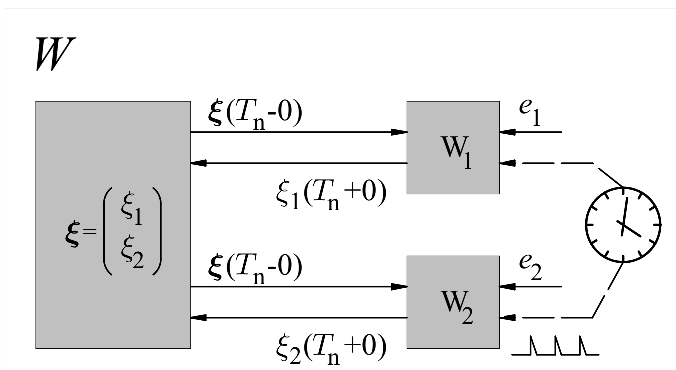

Figure 2: Two-component system

Fig. 2 reflects the fact that usually practical systems consist of several subsystems (components) the state of which are updated *synchronously* (say, when updating mechanism is triggered by a time impulse coming from an external clock). In this case equation (1) also takes the more detailed form

$$\begin{aligned} \xi_1(T_n + 0) &:= f_1(\xi(T_n - 0)), \\ \xi_2(T_n + 0) &:= f_2(\xi(T_n - 0)). \end{aligned} \tag{2}$$

Now one may try to go further and to consider even more realistic situation when different components of the system W on Fig. 2 are updated non-synchronously with each other. Such systems are plotted on Fig. 3 where the so called *phase-asynchronous* mode of updating is demonstrated and on Fig. 4 where the so called *frequency-asynchronous* mode of updating is demonstrated.

And now, one may formulate the key problem:

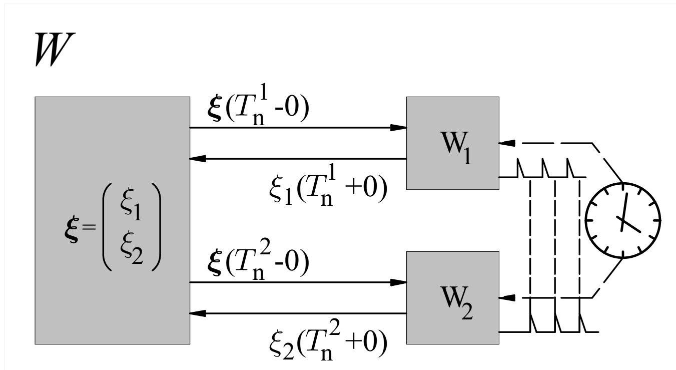

Figure 3: Two-component system with asynchronous updating moments $T_n^1 := n\tau + \varphi_1$, $T_n^2 := n\tau + \varphi_2$ where $\varphi_1 \neq \varphi_2$

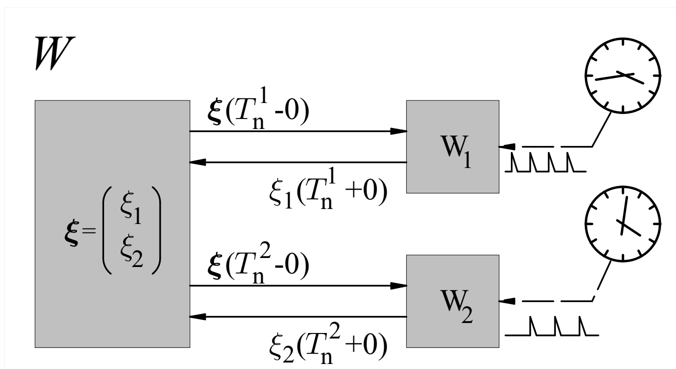

Figure 4: Two-component system with asynchronous updating moments: $T_n^1 := n\tau_1 + \varphi_1$, $T_n^2 := n\tau_2 + \varphi_2$ where $\tau_1 \neq \tau_2$

What happens with the dynamics of the system W if updating moments for different components do not coincide?

3 Asynchronous Systems

While for a two-component system with synchronous updating mode (*synchronous system*) we have the dynamic's equation (2), for a two-component system with asynchronous updating mode (*asynchronous system*) we obtain outwardly similar, but nevertheless different equations of dynamics:

$$
\begin{array}{rcl}
\xi_1(T_n^1+0) & := & f_1(\xi(T_n^1-0)), \\
\xi_2(T_n^2+0) & := & f_2(\xi(T_n^2-0)).
\end{array} \tag{3}
$$

This means, in fact, that for a given updating moment $t = T_n \in \{T_n^1\} \cup \{T_n^2\}$ the state vector $(\xi_1(t), \xi_2(t))$ of the system W is changed in accordance to the law

$$
\begin{array}{rcr}
\xi_1(T_n+0) & := & \xi_1(T_n-0), \\
\xi_2(T_n+0) & := & f_2(\xi(T_n-0)),
\end{array}
\quad \text{or} \quad
\begin{array}{cr}
:= & f_1(\xi(T_n-0)), \\
:= & \xi_2(T_n-0),
\end{array} \tag{4}
$$

depending on whether $T_n \in \{T_k^2\}$ while $T_n \notin \{T_k^1\}$ or vise versa $T_n \in \{T_k^1\}$ while $T_n \notin \{T_k^2\}$. And only in the case when $T_n = T_m^1 = T_k^2$ or, what is the same $T_n \in \{T_k^1\} \cap \{T_k^2\} \neq \emptyset$, updating law for the system W at the moment $t = T_n$ is described by equations (2).

Unfortunately, description of the dynamics of the system W in terms of 'impulse' equations (2) or (3) is not very convenient. Thus, it is natural to describe the dynamics of W in terms of more habitual difference equations.

Clearly, as was mentioned earlier, in the simplest situation the dynamics of synchronous system W is covered by the vector equation (1) which can be rewritten in the following form:

$$
x(n+1) := f(x(n)) = \begin{pmatrix} f_1(x(n)) \\ f_2(x(n)) \end{pmatrix}.
$$

For asynchronous system by analogy with (4) one can get:

$$
x(n+1) := \begin{pmatrix} x_1(n) \\ f_2(x(n)) \end{pmatrix} \quad \text{or} \quad := \begin{pmatrix} f_1(x(n)) \\ x_2(n) \end{pmatrix}. \tag{5}
$$

depending on whether $T_n \in \{T_k^2\}$ or $T_n \in \{T_k^1\}$.

4 Equations of Dynamics

Let us summarize now, what one needs to describe the dynamics of asynchronous system?

- For each n, one should know the set $\omega(n)$ of indices of the coordinates of the state vector x updated at the moment n. In the case of two-dimensional system the set $\omega(n)$ may be $\{1\}$ or $\{2\}$ or $\{1,2\}$; in the general case of N–component system the set $\omega(n)$ is a non-empty subset of the set $\{1, 2, \ldots, N\}$.

- For each set of indices ω, one should define the mapping $f_\omega(x)$ (*the ω–mixture of the mapping f*) i-th coordinate of which is defined in accordance with the rule

$$f_{\omega,i}(x) := \begin{cases} f_i(x), & \text{if} \quad i \in \omega, \\ x_i, & \text{if} \quad i \notin \omega; \end{cases} \tag{6}$$

- At last, one should write down the 'asynchronous' equation of dynamics

$$x(n+1) := f_{\omega(n)}(x(n)). \tag{7}$$

So, to describe the dynamics of asynchronous version of the system W one should replace 'synchronous' equation (1) by its 'asynchronous' counterpart (7).

As is seen, the right-hand term of 'asynchronous' equation (7) essentially depends on the procedure of 'taking the mixture' of the mapping f. So, look at some examples of mixtures for different mappings.

Example 4.1

$$\textit{Let} \quad f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \end{pmatrix}, \quad \textit{then} \quad f_{\{2\}}(x) = \begin{pmatrix} x_1 \\ f_2(x) \\ x_3 \\ x_4 \end{pmatrix}.$$

Example 4.2 *Let $f(x)$ be a linear mapping, $f(x) = Ax$, where A is the matrix*

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{44} \\ a_{21} & a_{22} & a_{23} & a_{44} \\ a_{31} & a_{32} & a_{33} & a_{44} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

then $f_\omega(x) = A_\omega x$ where

$$A_{\{2\}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{44} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad A_{\{1,3\}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 1 & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Another thing that we need to pay attention to when considering asynchronous equation (7) is the *index* sequence $\omega(n)$, as the properties of this sequence dramatically affect properties of the right-hand term of equation (7). These properties in more details will be discussed later, while now we shall make only one important remark.

Remark 4.3 From 'physical' considerations it is natural to suppose that each coordinate of the vector x is updated infinitely many times, i.e. each $i \in \{1, 2, \ldots, N\}$ belongs to infinitely many sets $\omega(n)$, $n \geq 0$ (the sequences $\omega(n)$ possessing this property will be called *admissible*). The meaning of such a requirement is that each component is 'alive' forever as in the opposite case the state vector of the corresponding component will be constant starting from some moment. When considering the long-term dynamics, this will allow to exclude this component from further considerations and to reduce the dimension of the state vector of the system under consideration.

5 Short Historical Survey

Now, when we have acquainted the reader with the notion of asynchronous system, we present a short survey of basic stages in forming the idea of asynchronous systems.

Long ago (we may say 'in prehistoric times') there were known the method of Simple Iterations and the method of Gauss–Seidel of solving linear or nonlinear equations. The method of Gauss–Seidel can be treated as the asynchronous version of the method of Simple Iterations with choosing

$$\omega(n) := \{n - 1 \pmod N + 1\}.$$

It is known (see, e.g., [1]) that the method of Simple Iterations and the method of Gauss–Seidel may converge or diverge independently of each other; thus, all combinations of stability/instability for the pair of synchronous system and its asynchronous counterpart may occur. In terms of asynchronous systems this leads to the principal conclusion: *stability/instability properties of a system may change dramatically depending on whether the system operates in synchronous or asynchronous mode.*

It seems that the first distinctive formulations of the idea of asynchronism, as applied to investigation of dynamic properties of control systems, have appeared somewhere in 50s. They were mainly connected with attempts to consider impulse systems in control theory with asynchronously interacting components (see, e.g., Sklansky & Raggaciny [25, 26], Kranc [19, 20], Fan Chung Wuy [13]). Principal attention was paid to investigation of the so-called 'multirate' impulse systems which are essentially (in our terminology) the systems with 'frequency' updating components. The main conclusion that may be drawn from these works is the fact that theoretical investigation of asynchronous systems, even in linear case, is much more complicated comparing with that for usual, synchronous, systems.

In 60-70s the main interest to asynchronous systems was motivated by necessity to develop tools for the so-called parallel methods of computation.

In 1969 the work of Chazan & Miranker [9] was published, in which the stability of linear asynchronous systems with positive matrices was 'fully' investigated. The word 'fully' is taken here in quotation marks because the principal necessary and sufficient condition of stability has been formulated in terms of 'all possible asynchronous systems with a given matrix', but not in terms of an individual asynchronous system with a given updating law $\{\omega(n)\}$.

In 90s an accumulated knowledge in the theory of asynchronous systems was summarized to some extent in monographs by Bertsekas & Tsitsiklis [3], Asarin, Kozyakin et al. [1] and Bhaya & Kaszkurewicz [4]. These three monographs precisely reflect three directions of investigation of asynchronous systems formed up to now. The first direction is originated from the needs of computational mathematics and its principal problem is how to organize computational procedure to obtain most efficiently (fast, with low memory consumption or processor load) converging iteration algorithm. The second direction is originated from considerations of control theory and its principal problem whether the system under consideration remains to be stable or unstable following to such a perturbation in its behavior as asynchronous data transmission between different its components. The third direction is the attempt to develop 'robust' linear algebraic conditions which enable stability of a system independently on whether this system operates in synchronous or asynchronous mode.

6 First Set of Problems

Let us formulate some natural problems arising in investigation of stability of asynchronous systems. It is worth to stress that, as it will be seen below, the main problems arising in investigation of asynchronous systems are originated not from the fact of linearity or non-linearity of the system under consideration. So, it what follows only linear systems will be considered.

Given an $N \times N$ real matrix A. Consider the usual 'synchronous' linear system

$$x(n+1) = Ax(n) \qquad (W)$$

and its asynchronous counterpart

$$x(n+1) = A_{\omega(n)}x(n) \qquad (W_a)$$

Question 6.1 *What kind of conditions should be imposed on A under which stability of (W) implies stability of (W_a)*

- *for all admissible (see definition in Remark 4.3) sequences $\{\omega(n)\}$?*
- *for sequences $\{\omega(n)\}$ from some class (corresponding, e.g., to phase or frequency updating mode, some stochastic low of updating of coordinates, etc.)?*
- *for some individual sequence $\{\omega(n)\}$?*

Another set of problems covers 'perturbation' properties of asynchronous equation (W_a). More precisely, given an $N \times N$ real matrix A and a sequence $\{\omega(n)\}$, such that the asynchronous equation (W_a) be asymptotically stable.

Question 6.2 *How will be changed the stability of (W_a) if we perturb the matrix A?*

Question 6.3 *How will be changed the stability of (W_a) if we 'slightly' perturb the updating sequence $\{\omega(n)\}$? And what is the meaning of the term 'slight perturbation of $\{\omega(n)\}$' from applicational point of view?*

In connection with the last question one should take into account that 'small' from the physical point of view perturbation of updating moments $\{T_n^i\}$ as a rule results in a 'big' perturbation of right-hand terms of asynchronous equations. For example, in the case of phase-frequency updating moments $T_n^i := n\tau_i + \varphi_i$ it is naturally from the physical point of view to treat perturbation of moments T_n^i as 'slight' or 'small' if the parameters τ_i or φ_i are slightly perturbed. But such a 'slight' perturbation of T_n^i will result in a rather 'big' in any reasonable metric perturbation of the set-valued sequence $\omega(n)$.

Even in physically 'natural' situations behavior of the sequence $\{\omega(n)\}$ is rather complicated, e.g. it is not periodic. As a result, it is more difficult to investigate 'individual' asynchronous equations than 'classes' of such equations. So, there naturally arises the following general question.

Question 6.4 *How the right-hand terms of asynchronous equations depend on $\omega(n)$?*

Another set of questions arise when one is interested in the problem of synthesis. Given an $N \times N$ real matrix A. Again consider the asynchronous equation (W_a).

Question 6.5 *Is it possible to choose such a sequence $\{\omega(n)\}$ which makes equation (W_a) stable?*

Question 6.6 *Is it possible to choose such a non-degenerate matrix Q and a sequence $\{\omega(n)\}$ which makes stable the 'equivalent' equation*

$$x(n+1) = \left(QAQ^{-1}\right)_{\omega(n)} x(n).$$

At last, formulate a very 'simple' question. Given an $N \times N$ real matrix A and equations

$$x(n+1) = Ax(n), \qquad (W)$$
$$x(n+1) = A^*x(n). \qquad (W^*)$$

It is well known that these equations are simultaneously either stable or unstable. Now, consider asynchronous versions of equations (W) and (W^*):

$$x(n+1) = A_{\omega(n)}x(n), \qquad (W_a)$$
$$x(n+1) = (A^*)_{\omega(n)}x(n). \qquad (W_a^*)$$

Question 6.7 *Is it valid that stability of equation (W_a) implies stability of equation (W_a^*) and vise versa?*

The questions formulated above are gathered here not by 'difficulty' principle but simply to demonstrate that even quite naturally formulated questions which have evident answers in 'synchronous' setting become not-so-evident when one start to investigate them in the 'asynchronous' formulation. In more details these and other questions are discussed in [1]. Some of them have quite natural answers that can be obtained relatively easy, other also have natural answers but prove the corresponding statements there is needed to develop a special technique, and at last, among these questions there are such that unresolved until now.

7 Chazan–Miranker theorem

Partially, answers to some questions formulated in the previous Section will be given below.

Theorem 7.1 (Chazan & Miranker, [9]) *Let $A = (a_{ij})$ be a real matrix with positive entries, $a_{ij} > 0$. If $\rho(A) < 1$ then any asynchronous equation (W_a) is asymptotically stable. If $\rho(A) \geq 1$ then such a sequence $\{\omega(n)\}$ can be found that the corresponding asynchronous equation will be not asymptotically stable.*

In the control theory, instead of words ‘any equation (from some class) is stable’, usually are said ‘equation is *absolutely stable in a class of all asynchronous equations* (W_a). This remark makes it possible to reformulate Theorem 7.1 in the following way.

Theorem 7.2 *Let $A = (a_{ij})$ be a real matrix with positive entries, $a_{ij} > 0$. Then the asynchronous equation (W_a) is absolutely asymptotically stable in the class of all admissible updating sequences $\{\omega(n)\}$ (see definition in Remark 4.3) if and only if $\rho(A) < 1$.*

The following theorem allows to use Chazan–Miranker criterium for obtaining sufficient conditions of stability of asynchronous systems with arbitrary matrices.

Theorem 7.3 (Majorization Principle) *Let $B = (b_{ij})$ be a real matrix with positive entries satisfying $\rho(B) < 1$, and let the matrix $A = (a_{ij})$ be such that $|a_{ij}| \leq b_{ij}$. Then the asynchronous equation (W_a) is absolutely asymptotically stable in the class of all admissible updating sequences $\{\omega(n)\}$.*

The following criterium of stability is again a word-to-word reformulation of the fact well known in the ‘synchronous’ setting.

Theorem 7.4 ([1]) *If the matrix A symmetric, $A = A^*$, then the asynchronous equation (W_a) absolutely asymptotically stable in the class of all admissible updating sequences $\{\omega(n)\}$ if and only if $\rho(A) < 1$.*

In [1] a criterium of absolute asymptotical stability of equation (W_a) in the class of all admissible updating sequences $\{\omega(n)\}$ for the case of 2×2 matrices A is obtained.

Unfortunately, theorems presented in this Section almost exhaust the set of easily formulated statements concerning investigation of the problem of absolute stability of asynchronous systems known up to date. Another frustrating thing is that proofs of the theorems formulated above are much more complicated than proofs of their ‘synchronous’ analogs.

The proof of ‘sufficient parts’ of the above theorems is based on the following remark: it suffices to find a norm $\|\cdot\|$ in $\mathbb{R}^N$ (called *joint strongly contracting norm*) such that

$$\|A_\omega\| \leq 1, \tag{8}$$

$$\|A_{\omega_1} A_{\omega_2} \cdots A_{\omega_k}\| \leq \gamma < 1, \qquad \text{as soon as} \quad \bigcup_{i=1}^{k} \omega_i = \{1, 2, \ldots, N\}, \tag{9}$$

in order to guarantee absolute asymptotic stability of the asynchronous equation (W_a) with the matrix A.

With this remark, the 'sufficient' part of Theorem 7.1 is obtained with the choice of the norm $\|x\| = \max_i\{\lambda_i|x_i|\}$ and appropriate λ_i, $i = 1, 2, \ldots, N$. To prove the 'sufficient' part of Theorem 7.4 it suffices to define the norm $\|x\|$ as $\|x\| = \sqrt{((I - A)x, x)}$. For the case $N = 2$ the definition of corresponding norm is more complicated.

What is important is that, in fact, conditions (8), (9) are not only sufficient for stability of asynchronous systems, but also necessary.

Theorem 7.5 ([1]) *Existence of a norm $\|\cdot\|$ satisfying condition* (8) *is necessary and sufficient for absolute stability of asynchronous equation* (W_a).

Existence of a norm $\|\cdot\|$ satisfying conditions (8), (9) *with some $\gamma < 1$ is necessary and sufficient absolute asymptotical stability of asynchronous equation* (W_a).

8 Complexity Issues

Note, that usual spectral criterium $\rho(A) < 1$ of asymptotic stability of the 'synchronous' equation (W), as is well known, equivalent to the condition

$$\exists \; \|\cdot\|, \gamma < 1 : \qquad \|A\| \leq \gamma < 1, \tag{10}$$

which is exactly the condition (9) presented in Theorem 7.5. The spectral condition $\rho(A) < 1$ can be treated as 'simple' from various points of view — it is algorithmic, it is semialgebraic in terms of entries of the matrix A, i.e., it can be written as a finite set of algebraic equalities and inequalities over entries of the matrix A, etc. So, condition (10) is also may be qualifies as 'simple', which gives us hope that conditions (8)–(9) are also rather 'simple' for use.

Unfortunately, this is not the case. Given a set of $N \times N$ matrices A_1, A_2,…,A_k. The norm $\|\cdot\|$ in $\mathbb{R}^N$ will be called *joint contracting (nonexpanding)* if

$$\|A_1\|, \|A_2\|, \ldots, \|A_k\| < 1 \quad (\leq 1).$$

Clearly, this condition is of the same type as conditions (8)–(9).

Theorem 8.1 (Kozyakin, [1]) *If $N, k \geq 2$ then the problem of existence of the joint contracting (nonexpanding) norm is not semialgebraic.*

Loosely speaking, Theorem 8.1 means that the problem of existence of the joint contracting (nonexpanding) norm for a set of matrices cannot be resolved by finite Boolean combinations of algebraic formulae.

Solutions to finite combinations of algebraic equalities and inequalities form semialgebraic sets with a variety of good (tame) properties which makes them a rather attractive object in mathematical constructions. To isolate the basic properties of semialgebraic sets, in the 80's Van den Dries, Knight, Pillay and Steinborn [10,14,24] axiomatically introduce the notion of o-minimal structures and prove that the main bulk of tame properties of semialgebraic sets are inherited by sets definable in o-minimal structures (see., e.g., [11]). Remark that analytical functions (seemingly the most natural candidate to inherit the basic properties of semialgebraic sets) does not form an o-minimal structure nor contained in any o-minimal structure.

One of the most broad and useful in practice examples of the sets, definable in o-minimal structures, is the family $\mathcal{S}_{\mathrm{an,exp}}$ [12] of all sets which can be described by finite Boolean combinations of formulae composing of finite number of algebraic operations, the operation of exponentiation and application of a finite number of the restricted analytical functions. Here by restricted analytical function is meant the function which is equal to an analytical function on some compact set and to zero outside of it.

In [16] a generalization of Theorem 8.1 is proved stating that the problem of existence of the joint contracting (nonexpanding) norm for a set of matrices is undefinable in o-minimal structures. Informally, this means that this problem is unsolvable by finite Boolean combinations of formulae composing of finite number of algebraic operations, the operation of exponentiation and application of a finite number of the restricted analytical functions.

The proofs in [16] (so as the proof of Theorem 8.1) are essentially based on two facts: that sets definable in o-minimal structures have only finitely many components of connectedness and that polynomial images of such sets are again sets definable in o-minimal structures (Tarski–Seidenberg principle).

Blondel & Tsitsiklis [3] show that analogous problem of computing and approximating the so-called joint spectral radius of a set of matrices A_1, $A_2,\ldots,A_k$ is NP-hard, when not impossible.

9 Robustness of Stability

In spite of the fact that Theorem 7.5, as indicated in the previous Section, is not too constructive, it nevertheless may help to derive quite strong properties of asynchronous systems (details see, e.g., in [1]).

Theorem 9.1 (Robustness of Stability) *Let asynchronous equation* (W_a) *be absolutely asymptotically stable and* B *be a matrix with sufficiently small* *norm $\|B\|$. Then the perturbed asynchronous equation*

$$x(n+1) = (A+B)_{\omega(n)}x(n)$$

is also absolutely asymptotically stable.

Theorem 9.2 (First Approximation Principle) *Let asynchronous equation (W_a) be absolutely asymptotically stable and function $f(x)$ satisfies 'first approximation condition' $f(x) = Ax + o(\|x\|)$. Then the zero equilibrium of nonlinear asynchronous equation*

$$x(n+1) = f_{\omega(n)}(x(n))$$

is absolutely asymptotically stable.

10 Finiteness Conjecture

For an asynchronous system

$$x(n+1) \quad = \quad A_{\omega(n)}x(n), \qquad (W_a)$$

one may pose the following questions.

Question 10.1 *How to characterize those index sequences $\{\omega(n)\}$ for which $\|x(n)\|$ tends to zero in a fastest way (or tends to infinity in a slowest way)?*

Question 10.2 *How to characterize those index sequences $\{\omega(n)\}$ for which $\|x(n)\|$ tends to zero in a slowest way (or tends to infinity in a fastest way)?*

First of the above questions arises in the problem of *stabilization* of a system (W_a) with the help of choice of appropriate desynchronization law $\{\omega(n)\}$. Second question is connected with the stability analysis of system (W_a). In an abstract setting Question 10.2 is closely related with the Finiteness Conjecture by Lagarias & Wang [21] formulated below.

Define, for a set of matrices $\mathcal{A} := \{A_1, A_2, \ldots, A_k\}$, the *greatest Lyapunov exponent* as

$$\lambda^+(\mathcal{A}) := \sup_{n\geq 1} \frac{1}{n} \log \left(\max_{A_{i_j}\in\mathcal{A}} \rho\left(A_{i_1}A_{i_2}\cdots A_{i_n}\right) \right)$$

Lagarias & Wang [21] conjectured that here sup may be replaced by max.

Recently, Bousch & Mairesse [8] built a counterexample to Finiteness Conjecture heavily based on measure theoretical considerations[1]. Unfortunately, their proof is provided in terms of the so-called topical maps, not matrices, which requires an additional effort to 'translate' their constructions on the 'matrix' language. Since constructions of Bousch & Mairesse [8] allows to get a partial answer to Question 10.2, below is given their geometric interpretation exploited some properties of discontinuous circle maps presented in [17, 18].

Let $A = (a_{ij})$ be a 2×2 matrix with positive entries, $a_{ij} > 0$. Then [2,8,27] there exist a special norm, called the *Barabanov* norm, $\|\cdot\|_b$ (where the lower index b stands for *Barabanov*) and a number $\lambda > 0$ such that[2]

$$\max\left\{\|A_{\{1\}}x\|_b, \|A_{\{2\}}x\|_b\right\} = \lambda\|x\|_b \quad \text{for} \quad x \geq 0.$$

It may be shown, that the set of those x for which $\|A_{\{1\}}x\|_b = \|A_{\{2\}}x\|_b$ is a straight line $L = \{x : \; l(x) = 0\}$, where $l(x)$ is a linear functional, crossing over the first and third quadrants. From this an 'explicit' construction of the fastest growing trajectory of asynchronous equation (W_a) can be readily derived. Provided that the point $x(n)$ ia already obtained, to construct $x(n+1)$ one should define it in the following manner

$$x(n+1) = F(x(n))$$

where the 'maximizing' map $F(x)$ is defined for $x = (x_1, x_2)$ with positive coordinates x_1 and x_2 as follows

$$F(x) = \begin{cases} A_{\{1\}}x, & \text{if} \quad l(x) \geq 0, \\ A_{\{2\}}x, & \text{if} \quad l(x) < 0. \end{cases}$$

From the definition of the maximizing map F it is clear that the sequence of vectors $\{x(n)\}$ will satisfy to asynchronous equation (W_a). The corresponding index sequence $\{\omega(n)\}$ is defined by the formula

$$\omega(n) = \begin{cases} \{1\}, & \text{if} \quad l(x(n)) \geq 0, \\ \{2\}, & \text{if} \quad l(x(n)) < 0. \end{cases}$$

As is shown in [8] the set-valued sequence $\omega(n)$ built above is *Sturmian*[3] (see, e.g., definition in [8, 17])[4]. Therefore [1], two-component asynchronous

[1] Later, another counterexample to the Finiteness Conjecture, which is exploited combinatorial properties of matrix products, was proposed by Blondel, Theys & Vladimirov [6,7].

[2] Idea of construction such a norm is taken from the works of Barabanov [2] and Mañé [22, 23], see also [27].

[3] More precisely, it is isomorphic to some Sturmian sequence.

[4] The same conclusion in a bit more constructive way may be drawn from geometric analysis of order preserving discontinuous circle maps [17, 18]

systems (W_a) which provide the the slowest convergence of solutions to zero (or fastest grows of solutions to infinity) are exactly those with frequency-asynchronous mode of updating.

11 Concluding Remarks

Of course, in a short paper it is impossible to give a detailed survey of all results obtained in the theory of asynchronous systems. So, below some of topics which would be interesting to discuss but which were skipped here due to lack of space are listed.

- Robustness of *instability* of asynchronous systems and its connection with the so-called overshooting effect and notion of quasi-controllability;
- problems specific to nonlinear asynchronous systems. Here, plenty of interesting and natural examples were found in the field of neural networks;
- to investigate stability of frequency asynchronous systems there were developed quite specific set of methods which are essentially based on the symbolic dynamics methods and especially on the properties of the so-called 'sturmian sequences'. Here, there are known a few robustness results (unfortunately, only for 2–component systems);
- stochastic asynchronous systems — in spite of very acute problems only some 'first' results are known;
- asynchronous analogs for differential equations — again, in spite of very natural statements of problems only some preliminary results were obtained in last years;
- from the applicational point of view it would be very important to develop approximate methods of investigation of individual asynchronous equations;
- etc.

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