Sturmian sequences generated by order preserving circle maps*

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There are plenty of known non-trivial properties of circle homeomorphisms. Sometimes, continuity of a circle map may be restrictive in applications. Therefore, the problem of distinguishing a class of circle maps retaining as much properties of homeomorphisms as possible while remaining rather broad is urgent. Clearly, by discarding continuity of a map one inevitably loses some properties inherent to circle homeomorphisms. Nevertheless, as it turned out discontinuous order preserving circle maps retain the majority of symbolical properties of circle homeomorphisms.

MSC 2000: 11B83, 37B10, 37E05, 37E10, 37E45, 68R15

Key words: Sturmian sequence, discontinuous map, order preserving map, circle map

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*This work was written while the author was a visitor of both Boole Centre for Research in Informatics and Department of Applied Mathematics, University College Cork, financial support from which was gratefully acknowledged. The work was also partially supported by grant No. 03-01-00258 of the Russian Foundation for Basic Research, by grant for Scientific Schools No. 1532.2003.1 of the President of Russian Federation and by the project "Development of mathematical methods of analysis of distributed asynchronous computational networks" of the program of fundamental investigations of OITVS RAN (Branch of Information Technologies and Computer Systems of Russian Academy of Sciences) "New physical and structural methods in infocommunications".

Contents

1 Introduction

Order preserving circle homeomorphisms possess plenty of interesting and non-trivial properties [6, 12] and play an important role in various fields of mathematics. Among such properties is the property of the rotation number of the homeomorphism f to be rational if and only if f has a periodic point [6, 12]. Another property that is perhaps less well known, is related to the so-called *Sturmian symbolical sequences* (see [11] and a preprint of a book under preparation [1]) associated with trajectories

$$x_{n+1} = f(x_n), \qquad n = 0, 1, \ldots. \tag{1}$$

Such symbolical sequences are found to be very useful in analysis of dynamical properties of various systems (see, e.g., [2, 3, 5, 7–9]).

However, continuity of a map f may sometimes be restrictive as is, e.g., in [2, Ch. VIII] or in [3]. So it is desirable to distinguish a class of circle maps retaining as many of the properties of homeomorphisms as possible whilst still being broad and containing both continuous and noncontinuous maps. One such class of maps will be considered below. It is the class of so-called order preserving circle maps which in general are not continuous.

Of course, if a circle map lacks such a strong attribute as continuity than it inevitably loses some of its properties. For example, a discontinuous circle map with rational rotation number may have no periodic points. Nevertheless, as it turns out iterations of points generated by such a map continue to produce Sturmian symbolic sequences.

The paper is organized as follows. In Section 2, we several equivalent definitions of Sturmian sequences are considered. In Section 3, basic properties of circle maps are discussed with emphasis on what happens when the continuity of the map is neglected and how the property of continuity is replaced. As such a replacement, in general discontinuous order preserving circle maps and their lifts, increasing maps of degree one, are chosen. In Section 4, it is shown that Sturmian sequences naturally arise in the symbolical description of trajectories of order preserving circle maps and their lifts. Here also "Sturmian coding" properties of the locally growing relaxation map of the interval $[0,1)$ into itself are investigated. This map appear in various applications. Section 5 contains concluding remarks on possible alternative ways to investigate properties of discontinuous circle maps.

2 Sturmian sequences

A binary symbolic sequence[1] $\boldsymbol{\sigma} = \sigma_0\sigma_1\ldots$ is called *Sturmian* if, for each $n = 1, 2, \ldots$, the number $p\ (n)$ of different subwords of the length n in this sequence equals to $n+1$. The function $p\ (n)$ is called also *the complexity function* of the sequence $\boldsymbol{\sigma}$.

Sturmian sequences play an important role in description of different phenomena. There are known different equivalent definitions of Sturmian sequences (see, e.g., [1, 11]), some of which will be presented below as they highlighted properties of Sturmian sequences from different points of view.

Cutting sequence. Consider the integer grid and a line with irrational slope in the plane (see Fig. 1), and build a sequence $\boldsymbol{\sigma}$ by writing down 0 or 1 each time the line intersects horizontal or vertical line of the grid, correspondingly. In the case when the line intersects a node of the grid one shell write down 0 or 1 at its own choice. The resulting sequence is called a *cutting sequence*. For example, the grid and line plotted on Fig. 1 generate the sequence $\boldsymbol{\sigma} = 0010010100100\ldots$. It is known, that *the sequence $\boldsymbol{\sigma}$ is cutting if and only if it is Sturmian.*

Rotation sequence. The sequence $\boldsymbol{\sigma} = \sigma_0\sigma_1\ldots$ is called a *rotation sequence* if there exist an irrational $\alpha \in [0,1]$ and a real x such that

$$\sigma_n = [(n+1)\alpha + x] - [n\alpha + x] \tag{2}$$

where $[t]$ denotes *the integral part* of a real number t, i.e., is the greatest integer does not exceeding t.

Rotation sequences occur in a number of natural situations, see e.g., [2, 5, 8, 9]. One of them is coding of trajectories of the circle shift map:

$$S : \mathbb{S}^1 \to \mathbb{S}^1 \qquad x \mapsto \{x + \alpha\}$$

where $\{t\} := t - [t] \equiv t \mod 1$ denotes *the fractional part of t*; clearly $\{t\} \in [0,1)$ for any t.

[1] A sequence is called *binary* if it consists of 0's and 1's.

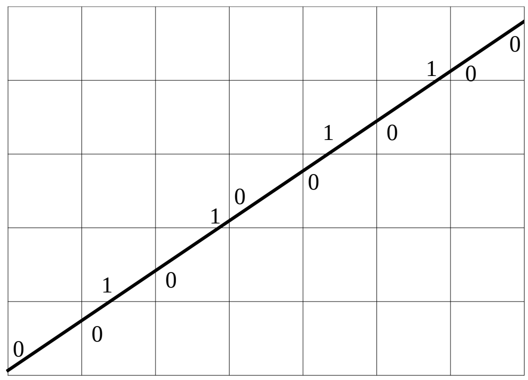

Figure 1: Cutting sequence

Divide $\mathbb{S}^1$ into two intervals $I_0 = [0, 1-\alpha)$ and $I_1 = [1-\alpha, 1)$, and denote by ν the *coding function* defined by

$$\nu(x) = \begin{cases} 0 & \text{if } x \in I_0, \\ 1 & \text{if } x \in I_1. \end{cases} \tag{3}$$

Then the rotation sequence $\boldsymbol{\sigma} = \sigma_0\sigma_1\ldots$ defined by α and x is just the sequence with $\sigma_n = \nu(S^n(x))$, see Fig. 2. It is known, that *the sequence* $\boldsymbol{\sigma}$ *is rotating if and only if it is Sturmian.*

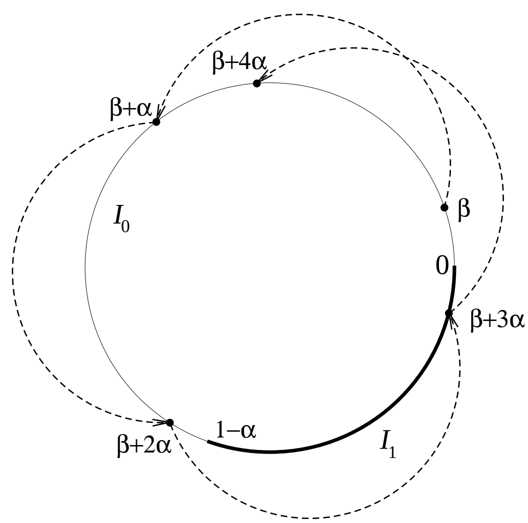

Figure 2: Rotation sequence

Balanced sequence. Consider the sequence $\boldsymbol{\sigma} = \sigma_0\sigma_1\ldots$ with $\sigma_n = 0, 1$ and denote by $|\boldsymbol{w}|_0$ or $|\boldsymbol{w}|_1$ the number of 0's or 1's respectively in the subword $\boldsymbol{w}$ of $\boldsymbol{\sigma}$. Then the sequence $\boldsymbol{\sigma}$ is called *balanced* if for any two its subwords $\boldsymbol{w}, \boldsymbol{v}$ of equal lengths the following is valid:

$$\big||\boldsymbol{w}|_0 - |\boldsymbol{v}|_0\big| \le 1 \quad \text{or, what is the same,} \quad \big||\boldsymbol{w}|_1 - |\boldsymbol{v}|_1\big| \le 1.$$

It is known, that *the ultimately non-periodic sequence $\boldsymbol{\sigma}$ is balanced if and only if it is Sturmian.*

In spite of simplicity of the definition, the structure of a Sturmian sequence is rather complicated. Mention some basic properties of such sequences, see, e.g., [1, 11]:

- *each Sturmian sequence $\boldsymbol{\sigma}$ is ultimately non-periodic (i.e., every its tail $\sigma_k\sigma_{k+1}\ldots$ is non-periodic);*
- *the frequency of 1's in a Sturmian sequence $\boldsymbol{\sigma}$, defined as the limit*

  $$\tau(\boldsymbol{\sigma}) = \lim_{n\to\infty} \frac{|\sigma_0\sigma_1\ldots\sigma_{n-1}|}{n}$$

  is well defined, and is irrational;
- *a Sturmian sequence is recurrent, that is, every word $\boldsymbol{w}$ that occurs in the sequence occurs infinite number of times with some well defined frequency $\tau(\boldsymbol{w})$.*

Some other properties of Sturmian sequences can be found in [1, 2, 5, 8, 9, 11].

Remark 1 The notions of Sturmian, cutting or rotating sequences, as they are given above, are related only to ultimately non-periodic sequences. Nevertheless, sometimes it is useful to consider broader classes of sequences with properties similar to properties of Sturmian sequences, except the property of ultimate non-periodicity.

For example, in the definition of Sturmian sequence one may consider the sequences complexity function of which satisfies $p\ (n) \le n+1$ instead of $p\ (n) \equiv n+1$. In the definition of cutting sequence one may consider all the sequences generated by lines with arbitrary slope (not only irrational)[2]. In the definition of rotation sequence one may consider all the sequences generated by the shift rotation map with arbitrary rotation angle α (not only irrational).

[2]Of course, in order that the notion of cutting sequence in this case be precise, we need to state what happens when at the nodes of the grid. In this case we should systematically take either 0 or 1, but not mix them.

3 Order preserving circle maps and their lifts

Denote by $\mathbb{S}^1 := \mathbb{R}/\mathbb{Z}$ the circle which is convenient to treat as the interval $[0, 1)$ with topologically identified points 0 and 1.

Pairwise distinctive points x, y, z from the circle $\mathbb{S}^1 = [0, 1)$ are said to have *natural cyclic ordering* if $(x-y)(y-z)(x-z) > 0$ and have *reverse cyclic ordering* if $(x-y)(y-z)(x-z) < 0$ (see detailed discussion of the notion of cyclic ordering in [4]). The map $f : \mathbb{S}^1 \to \mathbb{S}^1$ will be called *order preserving* if the cyclic ordering of the points $f(x), f(y), f(z)$ coincides with the cyclic ordering of the points x, y, z, i.e.,

$$\frac{f(x)-f(y)}{x-y} \cdot \frac{f(x)-f(z)}{x-z} \cdot \frac{f(y)-f(z)}{y-z} > 0, \tag{4}$$

as soon as points $x, y, z \in [0, 1)$ are pairwise distinctive. The following lemma gives a geometrical characterization of order preserving circle maps.

Lemma 1 *The map $f : \mathbb{S}^1 \to \mathbb{S}^1$ is order preserving if and only if there exist subintervals $I_+(f), I_-(f) \subseteq [0, 1)$, one of which may be empty, such that (see Fig.3)*

(i) $0 \in I_+(f)$, $I_+(f) \cap I_-(f) = \emptyset$, $I_+(f) \cup I_-(f) = [0, 1)$;

(ii) $f(x)$ is one-to-one increasing map on each of the intervals $I_+(f)$ and $I_-(f)$;

(iii) $f(x) > f(y)$ for any $x \in I_+(f)$, $y \in I_-(f)$.

PROOF. Remark first that in view of (4) the map f is one-to-one with its image, or injective. Define the sets $I_+(f)$ and $I_-(f)$ as follows:

$$I_+(f) := \{x \in [0, 1) : \ f(x) \geq f(0)\}, \quad I_-(f) := \{x \in [0, 1) : \ f(x) < f(0)\}.$$

By setting $z = 0$ in (4) we get

$$f(x) < f(y) \quad \text{if} \quad x < y \quad \text{and} \quad x, y \in I_+(f) \quad \text{or} \quad x, y \in I_-(f). \tag{5}$$

Show now that the sets $I_+(f), I_-(f)$ are intervals. Clearly $I_+(f) \cup I_-(f) = [0, 1)$ and $0 \in I_+(f)$, so it suffices to prove that $I_+(f)$ is interval. By supposing the contrary one can find $x, y, z \in [0, 1)$ such that $x < y < z$ and $x, z \in I_+(f)$, $y \in I_-(f)$. Then $(x-y)(y-z)(x-z) < 0$ while (5) implies

$$f(x) - f(y) > 0, \quad f(x) - f(z) < 0, \quad f(y) - f(z) < 0$$

which contradicts to (4). So, both the sets $I_+(f)$, $I_-(f)$ are intervals and properties (i)–(iii) are proved.

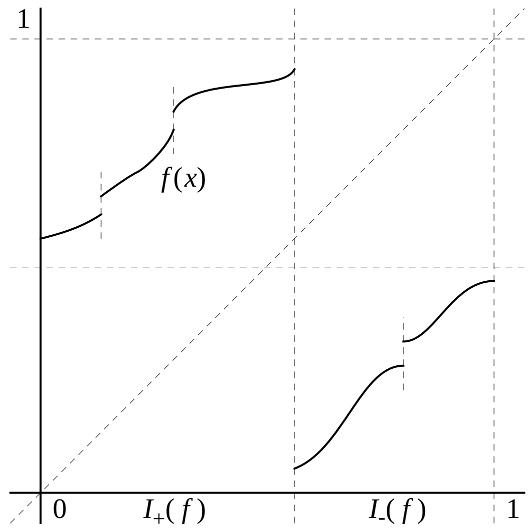

Figure 3: Order preserving circle map

At last, any map satisfying conditions (i)–(iii) is clearly order preserving as the map from $\mathbb{S}^1$ into itself. Lemma is proved. □

A convenient and prevalent tool to investigate properties of circle maps is the so-called lift of a map. The map $F : \mathbb{R} \to \mathbb{R}$ is called *the lift* of the circle map $f : \mathbb{S}^1 \to \mathbb{S}^1$ if $f = \{F\}$. As is easy to see for any lift of a circle map the value $F(x+1) - F(x)$ is integer. If the lift of a circle map is continuous then $F(x+1) - F(x) \equiv k$ with integer k; in this case k is called *degree* of F and denoted by $\deg(f)$, see, e.g., [6]. Clearly, lifts of orientation preserving circle homeomorphisms are maps of degree one. So, in what follows we will be interested primarily in lifts of degree one of circle maps.

The lift of a circle map is defined non-uniquely and to make this notion useful it is necessary to impose some additional properties on F. For example, when considering the lifts of circle homeomorphisms one usually requires also the continuity of the map F. In our case, when considering generally discontinuous order preserving circle maps we will be interested only in strictly monotone (increasing) lifts $F : \mathbb{R} \to \mathbb{R}$ of degree one, i.e., lifts satisfying

$$F(x+1) \equiv F(x) + 1, \qquad F(x) < F(y) \quad \text{for} \quad x < y. \tag{6}$$

If map F of degree one is strictly monotone then its restriction $f(x) = \{F(x)\}$ to the circle $\mathbb{S}^1$ is order preserving and is one-to-one with its image $f(\mathbb{S}^1)$. It is worth mentioning also that condition (6) implies

$$0 < F(y) - F(x) < 1 \quad \text{for} \quad 0 < y - x < 1. \tag{7}$$

Lemma 2 *For any order preserving circle map f there exists a strictly monotone lift of degree one. Any two strictly monotone lifts of f of degree one differ from one another by a constant.*

Proof. Let $I_+(f)$ and $I_-(f)$ be intervals mentioned in Lemma 1. Define the function $F(x)$ for $x \in [0,1)$ as

$$F(x) := \begin{cases} f(x) & \text{if} \quad x \in I_+(f), \\ f(x)+1 & \text{if} \quad x \in I_-(f). \end{cases}$$

and extend it on the whole set of reals $\mathbb{R}$ preserving "degree one" property. Clearly, such a function F will be increasing and have degree one.

Now, if F is strictly monotone map of degree one, then $f(x) = F(x) \bmod 1 \equiv \{F(x)\}$, $x \in [0,1)$, looks like the function plotted on Fig. 3. Then by Lemma 1 $f(x)$ is an order preserving circle map.

At last, let $F(x)$ and $G(x)$ be lifts of the same order preserving circle map $f(x)$, i.e.,

$$\{F(x)\} = \{G(x)\} = f(x).$$

Then $F(x)$ and $G(x)$ may be represented as

$$F(x) = m(x) + \{F(x)\}, \quad G(x) = n(x) + \{G(x)\}, \qquad x \in \mathbb{R},$$

where $m(x)$ and $n(x)$ are integer-valued functions. Hence

$$(m(x) - n(x)) - (m(y) - n(y)) = (F(x) - F(y)) - (G(x) - G(y)). \tag{8}$$

Here, since F and G are maps of degree one then

$$(F(x) - F(y)) - (G(x) - G(y)) = (F(s) - F(t)) - (G(s) - G(t))$$

where $s = \{x\} \in [0,1)$, $t = \{y\} \in [0,1)$. An then from (7) applied to both of the maps F and G we get

$$|(F(s) - F(t)) - (G(s) - G(t))| < 1. \tag{9}$$

So, (8)–(9) imply

$$|(m(x) - n(x)) - (m(y) - n(y))| < 1.$$

For the integer-valued function $m(x) - n(x)$ the latter inequality may be valid only in the case $m(x) - n(x) = \text{const}$ which means that $F(x) - G(x) = m(x) - n(x) \equiv \text{const}$. Lemma is proved. □

In what follows it will be useful the following

Lemma 3 *Any iteration of strictly monotone map F of degree one is also strictly monotone map of degree one. The map $F_*(x) = F(x) - x$ is 1-periodic and satisfies*

$$|F_*(x) - F_*(y)| < 1, \qquad \forall\, x, y \in \mathbb{R}. \tag{10}$$

Proof. The fact that n-th iteration of F is a strictly monotone map of degree one immediately follows by induction from the analogous properties of F.

The map $F_*(x) = F(x) - x$ is 1-periodic as the difference of two maps of degree one, $F(x)$ and x. To prove (10) note that the strict monotonicity of $F(x)$ implies the relations

$$0 < F_*(x) - F_*(y) + (x - y) < 1, \qquad 0 < x - y < 1.$$

Then $|F_*(x) - F_*(y)| < 1$ for $0 < x - y < 1$ from which, in view of already proven 1-periodicity of $F_*(x)$, the estimate (10) follows. □

4 Main results

In this Section a result which generalizes "Sturmian coding" properties of rotation sequences to sequences generated by a strict monotone (but generally discontinuous) map $F : \mathbb{R} \to \mathbb{R}$ of degree one will be formulated and proved.

Set $\alpha = \{F(0)\}$ and denote $I_1 = [0, \alpha)$, $I_0 = [\alpha, 1)$.[3]

Theorem 1 *Let I_0, I_1 be intervals defined above and ν be a coding function (3). Then the coding sequence $\boldsymbol{\sigma}(F, x) = \sigma_0(F, x)\sigma_1(F, x)\ldots$ with $\sigma_n(F, x) := \nu(\{F^n(x)\})$, $n = 0, 1, \ldots$, is balanced for any $x \in [0, 1)$.*

Proof. Remark first that the sequence $\boldsymbol{\sigma}(f, x) = \sigma_0(F, x)\sigma_1(F, x)\ldots$ for any $x \in [0, 1)$ can be defined similar to (2):

$$\sigma_n(F, x) = [F^{n+1}(x) - F(0)] - [F^n(x) - F(0)].$$

Hence given two words

$$\boldsymbol{w} = \sigma_0(F, x)\sigma_1(F, x)\ldots\sigma_n(F, x), \qquad \boldsymbol{w'} = \sigma_0(F, x')\sigma_1(F, x')\ldots\sigma_n(F, x')$$

one can write

$$\begin{aligned} |\boldsymbol{w}|_1 &= \sum_{k=0}^{n} \left([F^{k+1}(x) - F(0)] - [F^k(x) - F(0)]\right) = \\ &= [F^{n+1}(x) - F(0)] - [x - F(0)] \end{aligned} \tag{11}$$

[3]Remark, that in the case when $F(0)$ is integer and thus $\alpha = 0$ the set I_1 is empty while I_0 coincides with $[0, 1)$.

and analogously

$$
\begin{aligned}
|\boldsymbol{w'}|_1 &= \sum_{k=0}^{n} \big([F^{k+1}(x') - F(0)] - [F^{k}(x') - F(0)]\big) = \\
&= [F^{n+1}(x') - F(0)] - [x' - F(0)]. \qquad (12)
\end{aligned}
$$

Since F is strictly monotone function of degree one then one may suppose without loss in generality that $0 \le x \le x' < 1$. Then

$$
0 \le (x' - F(0)) - (x - F(0)) = x' - x < 1 \qquad (13)
$$

and by Lemma 3

$$
0 \le \big(F^{n+1}(x') - F(0)\big) - \big(F^{n+1}(x) - F(0)\big) = F^{n+1}(x') - F^{n+1}(x) < 1. \qquad (14)
$$

From (13) and (14) immediately follows that

$$
0 \le [x' - F(0)] - [x - F(0)] \le 1
$$

and

$$
0 \le [F^{n+1}(x') - F(0)] - [F^{n+1}(x) - F(0)] \le 1,
$$

and so

$$
\big| |\boldsymbol{w'}|_1 - |\boldsymbol{w}|_1 \big| \le 1.
$$

Theorem is proved. □

For lifts of circle homeomorphisms there is known (see, e.g., [6,12]) a very important characteristics, called *rotation number* $\tau(F)$ of the lift F (or of the corresponding homeomorphism):

$$
\tau(F) = \lim_{n\to\infty} \frac{F^n(x) - x}{n}.
$$

It is known that the function $\tau(F)$ is well defined, i.e., it exists and does not depend on x. Extension of the notion of rotation number to the case of generally discontinuous srtictly monotone maps of degree one can be found in [10].

From Theorem 1 immediately follows the corollary establishing connection between rotation number of the map F and frequency of the corresponding Sturmian sequence.

Corollary 1 *$\tau(\boldsymbol{\sigma}(F, x)) \equiv \tau(F)$.*

Proof. Fix $x \in [0,1)$ and set $\boldsymbol{\sigma} = \boldsymbol{\sigma}(F,x)$. Then from (11) it follows that

$$\tau(\boldsymbol{\sigma}(F,x)) = \lim_{n\to\infty} \frac{|\sigma_0\sigma_1\ldots\sigma_{n+1}|}{n+1} = \lim_{n\to\infty} \frac{[F^{n+1}(x) - F(0)] - [x - F(0)]}{n+1}.$$

But on the other hand, clearly

$$\lim_{n\to\infty} \frac{[F^{n+1}(x) - F(0)] - [x - F(0)]}{n+1} = \lim_{n\to\infty} \frac{F^{n+1}(x) - x}{n+1} = \tau(F),$$

which completes the proof. □

Now, results obtained above will be applied to investigate symbolical properties of a locally growing relaxation map arising in various applications, see, e.g., [2, Ch. VIII] and [3].

Let $f : [0,1) \to [0,1)$ be a function satisfying conditions

- $\exists \alpha \in (0,1)$ such that $f(x)$ is continuous and strictly increasing on the intervals $[0,\alpha)$ and $[\alpha,1)$;
- $x < f(x) < 1$ if $0 \le x < \alpha$;
- $0 \le f(x) < x$ if $\alpha \le x < 1$;
- $f(y) < f(x)$ if $0 \le x < \alpha \le y < 1$.

The function satisfying above conditions will be called *the locally growing relaxation function*. Typical graph of such a function is plotted on Fig. 4.

Consider, as in Section 2, two intervals $I_0 = [0,\alpha)$ and $I_1 = [\alpha,1)$ and denote by ν the coding function defined by

$$\nu(x) = \begin{cases} 0 & \text{if } x \in I_0, \\ 1 & \text{if } x \in I_1. \end{cases}$$

Theorem 2 *For any $x \in [0,1)$ the sequence $\boldsymbol{\sigma}_1(x) := \nu(f(x))\nu(f^2(x))\ldots$ is balanced.*[4]

Proof. Remark first that for any $x \in [0,1)$ the elements $f^n(x)$, $n \ge 1$, lay in the interval $[s,r)$ where

$$s := \inf_{\alpha \le x < 1} f(x), \qquad r := \sup_{0 \le x < \alpha} f(x).$$

[4] Remark that in the statement of Theorem 2 it is intentionally considered the 1-tail $\boldsymbol{\sigma}_1(x)$ of the full coding sequence $\boldsymbol{\sigma}(x) := \nu(x)\nu(f(x))\nu(f^2(x))\ldots$ generated by the function f and point x. Whether Theorem 2 is valid for the full coding sequence is a question.

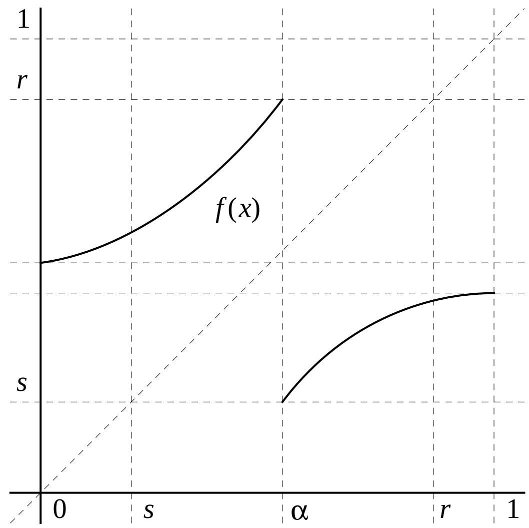

Figure 4: Locally growing relaxation function

Consider now two intervals $\tilde{I}_0 = [s, \alpha)$ and $\tilde{I}_1 = [\alpha, r)$ and denote by $\tilde{\nu}$ the coding function defined by

$$\tilde{\nu}(x) = \begin{cases} 0 & \text{if } x \in \tilde{I}_0, \\ 1 & \text{if } x \in \tilde{I}_1. \end{cases}$$

Then from inclusion $f^n(x) \in [s, r)$, $n \geq 1$, it follows that $\nu(f^n(x)) = \tilde{\nu}(f^n(x))$ for $n \geq 1$, and so

$$\boldsymbol{\sigma}_1(x) \equiv \tilde{\boldsymbol{\sigma}}(y) \tag{15}$$

where

$$\tilde{\boldsymbol{\sigma}}(y) = \tilde{\nu}(y)\tilde{\nu}(f(y))\tilde{\nu}(f^2(y))\ldots, \qquad y = f(x) \in [s, r).$$

Without loss of generality one can suppose that $s = 0$ and $r = 1$. Indeed, in the opposite case perform the change of variables $\tilde{x} = (x - s)/(r - s)$ and consider the function

$$\tilde{f}(\tilde{x}) := \frac{f((r - s)\tilde{x} + s) - s}{r - s}.$$

The function $\tilde{f}$ will possess all the properties of the function f with $s = 0$ and $r = 1$ and some α. So from now on it will be supposed that (see Fig. 5)

$$\inf_{\alpha \leq x < 1} f(x) = 0, \qquad \sup_{0 \leq x < \alpha} f(x) = 1.$$

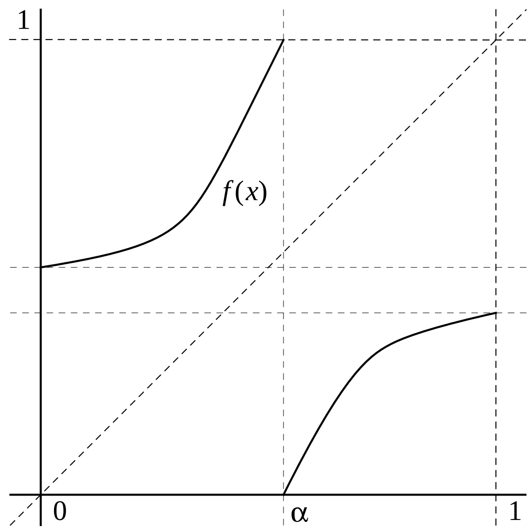

Figure 5: Function $f(x)$ with $s = 0$ and $r = 1$

Our goal is to use Theorem 1 to finalize the proof. But Theorem 1 deals with a strictly monotone map of degree one defined on $\mathbb{R}$ while in our case f is the map from $\mathbb{S}^1$ into itself. Besides, in Theorem 1 intervals $I_1 = [0, \alpha)$, $I_0 = [\alpha, 1)$ involved in construction of a coding sequence are defined in such a way that $\alpha = \{F(0)\}$ while in our case $f(\alpha) = 0$.

So, consider strictly monotone lift $F(x)$ of degree one of the function $f(x)$, which exists by Lemma 2. Then perform yet another change of variables to achieve $\alpha = \{F(0)\}$ by setting $\hat{x} = x - \alpha$. In new coordinates the function F will take the form $\hat{F}(\hat{x}) = F(\hat{x} + \alpha) - \alpha$ (see Fig. 6); clearly the function $\hat{F}$ is also strictly monotone function of degree one satisfying the condition $\{\hat{F}(0)\} = \hat{\alpha}$ with $\hat{\alpha} := \{-\alpha\} = 1 - \alpha$.

To finalize the proof consider again the point $y = \{F(x)\}$. Then introduce intervals $\hat{I}_0 = [\hat{\alpha}, 1)$ and $\hat{I}_1 = [0, \hat{\alpha})$ and denote by $\hat{\nu}$ the coding function defined by

$$\hat{\nu}(y) = \begin{cases} 0 & \text{if } y \in \hat{I}_0, \\ 1 & \text{if } y \in \hat{I}_1. \end{cases}$$

From the definition of the function $\hat{F}$ we get for $\hat{y} = y - \alpha$:

$$\{F^n(y)\} \in \tilde{I}_0 \Leftrightarrow \{\hat{F}^n(\hat{y})\} \in \hat{I}_0, \qquad \{F^n(y)\} \in \tilde{I}_1 \Leftrightarrow \{\hat{F}^n(\hat{y})\} \in \hat{I}_1.$$

Then $\tilde{\nu}(\{F^n(y)\}) \equiv \hat{\nu}(\{\hat{F}^n(\hat{y})\})$, and so

$$\hat{\boldsymbol{\sigma}}(\hat{y}) \equiv \tilde{\boldsymbol{\sigma}}(y) \tag{16}$$

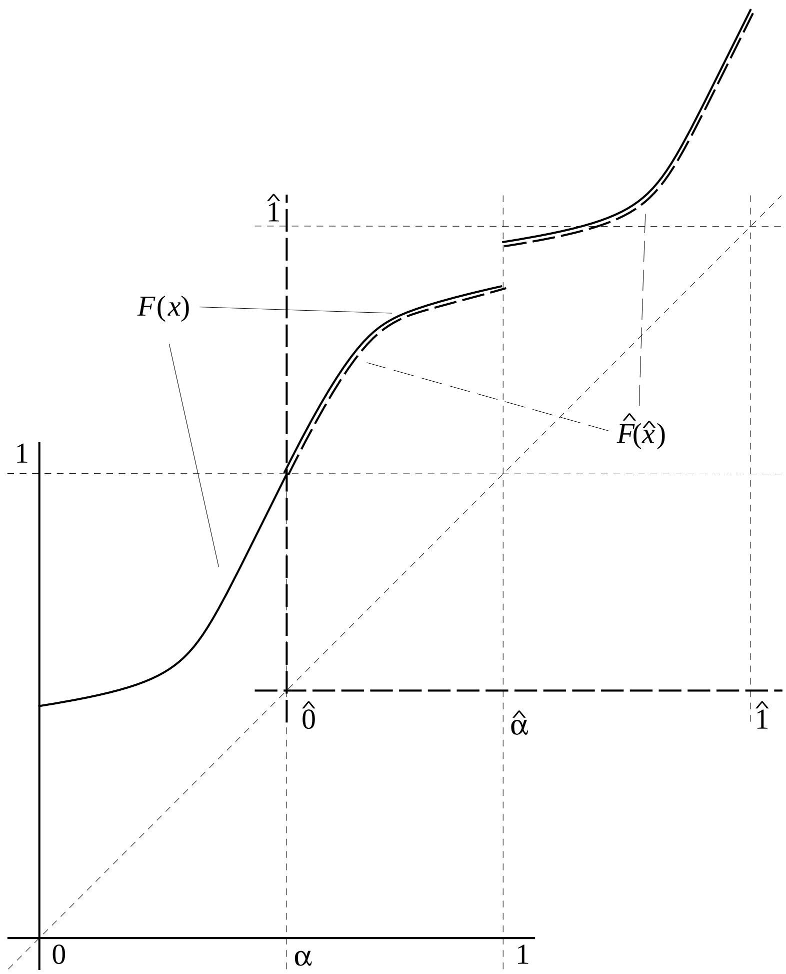

Figure 6: Functions $F(x)$ and $\hat{F}(\hat{x})$

where

$$\hat{\boldsymbol{\sigma}}(\hat{y}) = \hat{\nu}(\hat{y})\hat{\nu}(\hat{F}(\hat{y}))\tilde{\nu}(\hat{F}^2(\hat{y}))\ldots.$$

Now, the function $\hat{F}$ is subjected to Theorem 1 and thus the sequence $\hat{\boldsymbol{\sigma}}(\hat{y})$ is balanced. Then by (16) the sequences $\tilde{\boldsymbol{\sigma}}(y)$ is also balanced and by (15) so is the sequence $\boldsymbol{\sigma}_1(x)$. Theorem is proved. □

5 Concluding remarks

Sometimes, it might be fruitful to rest an investigation of order preserving discontinuous maps on ideas of limiting behavior of the corresponding dynamical system (1). As is shown in [10] any order preserving (possibly discontinuous) circle map $f(x)$ with irrational rotation number $\tau(f)$ is semi-conjugate to circle shift $\rho_{\tau(f)}(x) := x + \tau(f) \mod 1$. From this, in the case when $\tau(f)$ is irrational, it is possible to derive almost all the basic properties of order preserving circle maps presented above. Yet, in the paper a direct way of analyzing properties of f is intentionally chosen since it allows to study f for arbitrary values of $\tau(f)$.

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