## TWENTY FOURTH IRISH MATHEMATICAL OLYMPIAD

Saturday, 7 May 2011

Second Paper

# Problems and Solutions

6. Prove that
$$\frac{2}{3}+\frac{4}{5}+\frac{6}{7}+\cdots+\frac{2010}{2011}$$
is not an integer.

**Solution** (Proposed by Gordon Lessells)

Let $S$ be the sum. Then
$$1005-S=\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots+\frac{1}{2011}=T\ .$$
Then $S$ is an integer if and only if $T$ is an integer. 2011 is prime. Let $M = 3\cdot 5\cdot 7\cdot 9\cdots 2009$. If $T$ is an integer then $MT$ is an integer.
$$MT=\frac{M}{3}+\frac{M}{5}+\frac{M}{7}+\cdots+\frac{M}{2009}+\frac{M}{2011}\ .$$
Each term is an integer except the last one which is not since 2011 is prime. Hence $T$ is not an integer.

7. In a tournament with $N$ players, $N < 10$, each player plays once against each other player scoring 1 point for a win and 0 points for a loss. Draws do not occur. In a particular tournament only one player ended with an odd number of points and was ranked fourth. Determine whether or not this is possible. If so, how many wins did the player have?

   **Solution** (Proposed by Gordon Lessells)

   If only one player has an odd score, the total number of points won must be odd. This leaves two possibilities, $N = 6$ and $N = 7$. Consider first the case $N = 6$. The player with an odd score must have won 1,3 or 5 games. If 5, the player would have come first. If 1, the player would have come 5th or 6th. Thus the odd score was 3. The three players ranked higher must have won 4 matches leaving 0 for the remaining two players which is impossible.

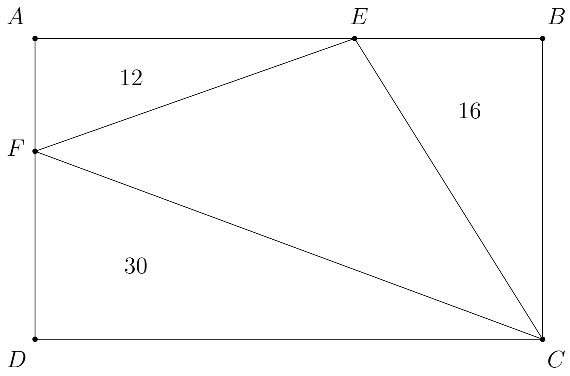
   Hence $N = 7$ is the only possibility. The score of the player with the odd score must be 1,3 or 5. 1 is impossible as the players ranked 5,6,7 must all have scored 0 in this case. Similarly, 5 is impossible for a player ranked fourth as the players ranked 1,2,3 must all have scored 6 in this case. It remains to show that 3 is a possibility. The players ranked 5,6,7 must have scored either 2,2,2 points or 2,2,0. In the first case, the first 3 players all score 4 points. In the second case, the first three score 6,4,4. Thus two possibilities occur: scores of 6,4,4,3,2,2,0 or 4,4,4,3,2,2,2. Both of these outcomes can be realised as shown below; hence the answer to the problem is 3.

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | Total |
|---|---|---|---|---|---|---|---|---|
| 1 | $X$ | 1 | 1 | 1 | 1 | 1 | 1 | 6 |
| 2 | 0 | $X$ | 1 | 1 | 1 | 0 | 1 | 4 |
| 3 | 0 | 0 | $X$ | 1 | 1 | 1 | 1 | 4 |
| 4 | 0 | 0 | 0 | $X$ | 1 | 1 | 1 | 3 |
| 5 | 0 | 0 | 0 | 0 | $X$ | 1 | 1 | 2 |
| 6 | 0 | 1 | 0 | 0 | 0 | $X$ | 1 | 2 |
| 7 | 0 | 0 | 0 | 0 | 0 | 0 | $X$ | 0 |

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | Total |
|---|---|---|---|---|---|---|---|---|
| 1 | $X$ | 1 | 0 | 1 | 1 | 0 | 1 | 4 |
| 2 | 0 | $X$ | 1 | 1 | 1 | 1 | 0 | 4 |
| 3 | 1 | 0 | $X$ | 1 | 1 | 0 | 1 | 4 |
| 4 | 0 | 0 | 0 | $X$ | 1 | 1 | 1 | 3 |
| 5 | 0 | 0 | 0 | 0 | $X$ | 1 | 1 | 2 |
| 6 | 1 | 0 | 1 | 0 | 0 | $X$ | 0 | 2 |
| 7 | 0 | 1 | 0 | 0 | 0 | 1 | $X$ | 2 |

8. $ABCD$ is a rectangle. $E$ is a point on $AB$ between $A$ and $B$, and $F$ is a point on $AD$ between $A$ and $D$. The area of the triangle $EBC$ is 16, the area of the triangle $EAF$ is 12 and the area of the triangle $FDC$ is 30. Find the area of the triangle $EFC$.

**Solution** (Proposed by Jim Leahy)




Let $|EB| = x$, $|AE| = y$. Then

$$|BC| = \frac{32}{x},\ |AF| = \frac{24}{y},\ |FD| = \frac{32}{x} - \frac{24}{y},\ |DC| = x + y\ .$$

Let $T$ denote the area of triangle $EFC$. Area of triangle $FDC = 30$. Therefore,

$$\frac{1}{2}\left(\frac{32}{x} - \frac{24}{y}\right)(x + y) = 30\ . \qquad (*)$$

Also, the area of $ABCD$ is

$$16 + 12 + 30 + T = \frac{32}{x}(x + y)$$

i.e.,

$$58 + T = \frac{32}{x}(x + y)$$

and so

$$T = \frac{32}{x}(x + y) - 58 = 32\left(1 + \frac{y}{x}\right) - 58\ . \qquad (**)$$

From (*), we have $(\frac{16}{x} - \frac{12}{y})(x + y) = 30$. Multiply across by $xy$:

$$\begin{aligned} &\Longrightarrow \quad (16y - 12x)(x + y) = 30xy \\ &\Longrightarrow \quad 16xy + 16y^2 - 12x^2 - 12xy = 30xy \\ &\Longrightarrow \quad 16y^2 - 26xy - 12x^2 = 0\ . \end{aligned}$$

Divide across by $x^2$:

$$\begin{aligned}
\implies & \quad 16\left(\frac{y}{x}\right)^2 - 26\left(\frac{y}{x}\right) - 12 = 0 \\
\implies & \quad \frac{y}{x} = \frac{26 \pm \sqrt{26^2 + 4 \cdot 16 \cdot 12}}{32} = \frac{26 \pm 38}{32} \\
\implies & \quad \frac{y}{x} = \frac{64}{32} = 2 \; .
\end{aligned}$$

Then, (**) gives

$$\begin{aligned}
T &= 32\left(1 + \frac{y}{x}\right) - 58 \\
&= 32(1+2) - 58 \\
&= 38 \; .
\end{aligned}$$

9. Suppose $x, y$ and $z$ are positive numbers such that

$$1 = 2xyz + xy + yz + zx. \tag{1}$$

Prove that

(i)

$$\frac{3}{4} \leq xy + yz + zx < 1;$$

(ii)

$$xyz \leq \frac{1}{8}.$$

Using (i) or otherwise, deduce that

$$x + y + z \geq \frac{3}{2}, \tag{2}$$

and derive the case of equality in (2).

**Solution** (Proposed by Finbarr Holland)

Since, by hypothesis, $xyz > 0$, the RHS of (i) follows from (1) immediately. Since it's clear that the LHS of (i) is equivalent to (ii), students may attack either (i) or (ii) and then deduce the other.

We offer two ways to prove the LHS of (i).

*First approach.* Suppose that the LHS of (i) is false for some triple of positive numbers $a, b, c$ that satisfy (1), so that $1 = 2abc + ab + bc + ca$, but $ab + bc + ca < 3/4$. Then, by the AM-GM inequality,

$$abc = \sqrt{(ab)(bc)(ca)} = \left(\sqrt[3]{(ab)(bc)(ca)}\right)^{3/2} \leq \left(\frac{ab + bc + ca}{3}\right)^{3/2} < \left(\frac{1}{4}\right)^{3/2} = \frac{1}{8},$$

whence

$$2abc + ab + bc + ca < \frac{2}{8} + \frac{3}{4} = 1,$$

in contradiction to our assumption. Hence, (i) holds, and (ii) now follows as an immediate consequence, because from (i) and (1) we see that

$$1 = 2xyz + xy + yz + zx \geq 2xyz + \frac{3}{4} \Leftrightarrow \frac{1}{4} \geq 2xyz \Leftrightarrow \frac{1}{8} \geq xyz.$$

*Second approach.* The proof of the LHS of (i) can be done directly as follows. By the AM-GM,

$$xyz = \left(\sqrt[3]{(xy)(yz)(zx)}\right)^{3/2} \leq \left(\frac{xy + yz + zx}{3}\right)^{3/2},$$

and so, with $t = (xy + yz + zx)/3$, we have that

$$1 = 2xyz + xy + yz + zx \le 2\big(t\big)^{3/2} + 3t \;\Leftrightarrow\; 1 \le 2(\sqrt{t})^3 + 3(\sqrt{t})^2,$$

i.e., letting $s = \sqrt{t}$, $r = 2s$

$$1 \le 2s^3 + 3s^2 \;\Leftrightarrow\; 4 \le r^3 + 3r^2 \;\Leftrightarrow\; 0 \le (r-1)(r+2)^2 \;\Leftrightarrow\; r \ge 1.$$

Thus, $t \ge 1/4$, i.e., $xy + yz + zx \ge 3/4$, as claimed.

We offer a direct way to prove (ii). By the AM-GM,

$$xy + yz + zx \ge 3\sqrt[3]{(xy)(yz)(zx)} = 3(xyz)^{2/3} = 3s^2 \text{ (where } s = \sqrt[3]{xyz}).$$

Hence

$$1 = 2xyz + xy + yz + zx \ge 2s^3 + 3s^2 \;\Leftrightarrow\; 0 \ge 2(s^3+1) + 3(s^2-1) = (s+1)^2(2s-1) \;\Leftrightarrow\; 2s \le 1.$$

Equivalently, $xyz = s^3 \le 1/8$. Again, (i) follows from this.

To prove (2), note that

$$(x + y + z)^2 \ge 3(xy + yz + zx) \ge \frac{9}{4} \;\Leftrightarrow\; x + y + z \ge \frac{3}{2},$$

with equality iff

$$\frac{9}{4} = (x + y + z)^2 = 3(xy + yz + zx),$$

i.e., $x + y + z = \frac{3}{2}$, and $x = y = z$, i.e., $x = y = z = 1/2$.

10. Find with proof all solutions in nonnegative integers $a, b, c, d$ of the equation

$$11^a 5^b - 3^c 2^d = 1.$$

**Solution** (Proposed by Tom Laffey)

We consider four cases: $d = 0$, $d = 1$, $d = 2$, $d > 2$.

*Case 1:* $d = 0$. In this case $3^c 2^d$ is odd, so there are no solutions.

*Case 2:* $d = 1$. We have $11^a 5^b - 2 \cdot 3^c = 1$, so $c > 0$, and, reading mod 4, we see that $a$ is odd. Next reading mod 3, we get that $a + b$ is even. Hence $a = 2\alpha + 1$, $b = 2\beta + 1$, for some nonnegative integers $\alpha, \beta$. Then $55 \cdot 11^{2\alpha} 5^{2\beta} - 2 \cdot 3^c = 1$ and $c \geq 3$. Suppose that $c > 3$. Then we have $55(11^{2\alpha} 5^{2\beta} - 1) = 54(3^{c-3} - 1)$. Hence $3^{c-3} - 1$ is divisible by 11. Since 5 is the order of 3 mod 11, $c - 3 = 5\gamma$, for some nonnegative integer $\gamma$, and $3^{c-3} - 1$ is divisible by $3^5 - 1 = 2 \cdot 11^2$. This contradicts the fact that $11^2$ does not divide $55(11^{2\alpha} 5^{2\beta} - 1) = 54(3^{c-3} - 1)$. So, in this case $a = 1$, $b = 1$, $c = 3$, $d = 1$ is the only solution.

*Case 3:* $d = 2$. Here $11^a 5^b - 4 \cdot 3^c = 1$, so reading mod 4, we see that $a$ is even. If $c = 0$, then we have the solution $a = 0$, $b = 1$, $c = 0$, $d = 2$. Suppose that $c > 0$. Reading mod 3, we see that $a + b$ is even, so $a$ is even and $b$ is even. Write $a = 2\alpha$, $b = 2\beta$. Now we have $(11^\alpha 5^\beta - 1)(11^\alpha 5^\beta + 1) = 4 \cdot 3^c$ and this is impossible since the left-hand side is divisible by 8.

*Case 4:* $d > 2$. Here $11^a 5^b - 3^c 2^d = 1$, and $d > 2$. Reading mod 4, we see that $a$ is even. If $c = 0$, then reading mod 8, we find that $b$ is even and thus, $a = 2\alpha$ and $b = 2\beta$ for some nonnegative integers $\alpha, \beta$. Hence $(11^\alpha 5^\beta - 1)(11^\alpha 5^\beta + 1) = 2^d$. Since $\gcd(11^\alpha 5^\beta - 1, 11^\alpha 5^\beta + 1) = 2$, and $11^\alpha 5^\beta - 1 > 2$, this is impossible. Hence $c > 0$. Reading mod 3, we see that $b$ is even, since $a$ is even. Writing $a = 2\alpha$, $b = 2\beta$, as before, we have $(11^\alpha 5^\beta - 1)(11^\alpha 5^\beta + 1) = 3^c \cdot 2^d$, so we have two possibilities: (i) $11^\alpha 5^\beta - 1 = 3^c \cdot 2$ and $11^\alpha 5^\beta + 1 = 2^{d-1}$, and (ii) $11^\alpha 5^\beta - 1 = 2^{d-1}$ and $11^\alpha 5^\beta + 1 = 3^c \cdot 2$, where we again use $\gcd(11^\alpha 5^\beta - 1, 11^\alpha 5^\beta + 1) = 2$. In case (i), we obtain, by subtraction, $2^{d-1} = 3^c \cdot 2 + 2$ and thus $2^{d-2} = 3^c + 1$. So $2^{d-2} - 1 = 3^c$ and $d - 2$ is even, say $d - 2 = 2\delta$ and then $(2^\delta - 1)(2^\delta + 1) = 3^c$ and thus $\delta = 1$ and $c = 1$, since $\gcd(2^\delta - 1, 2^\delta + 1) = 1$. But $11^\alpha 5^\beta - 1 = 6$ has no solutions. In case (ii), we have by subtraction $2 = 3^c \cdot 2 - 2^{d-1}$ and thus $2^{d-2} = 3^c - 1$. Then either $d = 3$ and $c = 1$ or $c = 2\gamma$ is even. For $c > 1$, we have $2^{d-2} = (3^\gamma - 1)(3^\gamma + 1)$ and $\gcd(3^\gamma - 1, 3^\gamma + 1) = 2$ and $\gamma = 1$ and $d = 5$. The case $c = 1$ gives the solution $a = 0$, $b = 2$, $c = 1$, $d = 3$. The case $c = 2$, $d = 5$ does not yield solutions.

Combining all cases, we see that the solutions are $11^1 \cdot 5^1 - 3^3 \cdot 2^1 = 1$, $11^0 \cdot 5^1 - 3^0 \cdot 2^2 = 1$ and $11^0 \cdot 5^2 - 3^1 \cdot 2^3 = 1$.