TWENTY FOURTH IRISH MATHEMATICAL OLYMPIAD

Saturday, 7 May 2011

First Paper

Problems and Solutions

1. Suppose $abc \neq 0$. Express in terms of a, b, and c, the solutions x, y, z, u, v, w of the equations

$$x + y = a, \quad z + u = b, \quad v + w = c, \quad ay = bz, \quad ub = cv, \quad wc = ax\,.$$

Solution (Proposed by Finbarr Holland)

Label the equations $1, 2, 3, 4, 5, 6$ in the order of their appearance. Using equations $3, 6$ eliminate w from the system, thereby adding to equations $1, 2, 4, 5$ the equation 7: $ax + cv = c^2$. Next, use this and 5 to eliminate v to produce equation 8: $ax + bu = c^2$. Eliminate z from $2, 4$ to produce equation 9: $ay + bu = b^2$. Now eliminate u from equations $8, 9$ giving $ax - ay = c^2 - b^2$. Finally, using this in conjunction with 1, we see that

$$2ax = c^2 + a^2 - b^2, \quad 2ay = a^2 + b^2 - c^2.$$

And so, too,

$$2cw = c^2 + a^2 - b^2, \quad 2bz = a^2 + b^2 - c^2.$$

Hence, using equations $2, 5$, we get that

$$2cv = 2bu = 2b^2 - 2bz = b^2 + c^2 - a^2.$$

Thus, letting

$$\alpha = \frac{b^2 + c^2 - a^2}{2bc}, \quad \beta = \frac{c^2 + a^2 - b^2}{2ca}, \quad \gamma = \frac{a^2 + b^2 - c^2}{2ab},$$

we see that

$$x = c\beta, \quad y = b\gamma, \quad z = a\gamma, \quad u = c\alpha, \quad v = b\alpha, \quad w = a\beta.$$

2. Let ABC be a triangle whose side lengths are, as usual, denoted by $a = |BC|$, $b = |CA|, c = |AB|$. Denote by m_a, m_b, m_c, respectively, the lengths of the medians which connect A, B, C, respectively, with the centres of the corresponding opposite sides.

   (a) Prove that $2m_a < b + c$. Deduce that $m_a + m_b + m_c < a + b + c$.

   (b) Give an example of

      (i) a triangle in which $m_a > \sqrt{bc}$;

      (ii) a triangle in which $m_a \le \sqrt{bc}$.

Solution (Proposed by Finbarr Holland)

Denote by D the mid-point of BC. We offer two ways of doing part (a).

First way: Continue the line segment AD through D to the point A' chosen so that $|A'D| = |DA| = m_a$. Consider the triangles $A'DC$ and ADB. Note that $\angle A'DC = \angle ADB$, $|BD| = |DC|$ and $|A'D| = |AD|$, by construction. Hence, these triangles are congruent and so, in particular, $|A'C| = |AB|$. Now consider the triangle $A'CA$, and apply the triangle inequality to infer that

$$2m_a = |A'D| + |DA| = |A'A| < |CA| + |A'C| = b + c,$$

i.e., (a) holds.

Second way: Two applications of the Cosine Rule tell us that

$$am_a \cos \angle BDA = m_a^2 + \left(\frac{a}{2}\right)^2 - c^2 \quad \text{and}$$
$$-am_a \cos \angle BDA = am_a \cos \angle CDA = m_a^2 + \left(\frac{a}{2}\right)^2 - b^2.$$

Thus, eliminating $\cos \angle BDA$, we see that

$$4m_a^2 = 2(b^2 + c^2) - a^2.$$

Hence $2m_a < b + c$ iff

$$2(b^2 + b^2) - a^2 < b^2 + 2bc + c^2 \iff b^2 + c^2 - a^2 < 2bc \iff \cos A < 1,$$

which is true.

In like manner,

$$2m_b < c + a, \quad 2m_c < a + b,$$

and so

$$2m_a+2m_b+2m_c < (a+b)+(b+c)+(c+a) = 2(a+b+c), \ \ m_a+m_b+m_c < a+b+c.$$

This completes the proof of part (a).

Part (b): The numbers $2, 4, 5$ are the side lengths of a triangle ABC with $a = 4, b = 2, c = 5$ for which

$$m_a^2 = \frac{2(b^2 + c^2) - a^2}{4} = \frac{21}{2} > 10 = bc.$$

Hence (i). (ii) occurs in any triangle in which $b = c$, because, in such a case,

$$bc = b^2 = m_a^2 + \left(\frac{a}{2}\right)^2 > m_a^2.$$

3. The integers $a_0, a_1, a_2, a_3, \ldots$ are defined as follows:

$$a_0 = 1, \quad a_1 = 3, \quad \text{and} \quad a_{n+1} = a_n + a_{n-1} \quad \text{for all } n \geq 1.$$

Find all integers $n \geq 1$ for which $na_{n+1} + a_n$ and $na_n + a_{n-1}$ share a common factor greater than 1.

Solution (Proposed by Bernd Kreussler)

Define $b_n(x) = a_{n+1}x + a_n$ for $n \geq 0$ and any integer x. We have to find all $n \geq 1$ for which $\gcd(b_n(n), b_{n-1}(n)) > 1$. By definition, for $k \geq 1$ we have $b_{k+1}(x) = b_k(x) + b_{k-1}(x)$, hence $\gcd(b_{k+1}(x), b_k(x)) = \gcd(b_k(x), b_{k-1}(x))$ and so, using induction and substituting $x = n$,

$$\begin{aligned} \gcd(b_n(n), b_{n-1}(n)) &= \gcd(b_1(n), b_0(n)) = \gcd(a_2 n + a_1, a_1 n + a_0) \\ &= \gcd(4n + 3, 3n + 1) = \gcd(n + 2, 3n + 1) \\ &= \gcd(n + 2, -5) \in \{1, 5\}. \end{aligned}$$

Hence, $na_{n+1} + a_n$ and $na_n + a_{n-1}$ are not coprime iff $5 \mid n+2$, i.e. $n \equiv 3 \pmod 5$.

4. The incircle $\mathcal{C}_1$ of triangle ABC touches the sides AB and AC at the points D and E, respectively. The incircle $\mathcal{C}_2$ of the triangle ADE touches the sides AB and AC at the points P and Q, and intersects the circle $\mathcal{C}_1$ at the points M and N. Prove that

   (a) the centre of the circle $\mathcal{C}_2$ lies on the circle $\mathcal{C}_1$;

   (b) the four points M, N, P, Q in appropriate order form a rectangle if and only if twice the radius of $\mathcal{C}_1$ is three times the radius of $\mathcal{C}_2$.

Solution (Proposed by Anca Mustata)

(a) Let I be the incentre of $\triangle ABC$, and let J be the point of intersection of AI and $\mathcal{C}_1$. As $\triangle ADI \equiv \triangle AEI$, the line AJ is the angle bisector of $\angle DAE$.

Also, the arcs DJ and EJ have the same measure. The measure of the angle $\angle ADJ$ is half of the arc DJ and the measure of $\angle JDE$ is half of the arc EJ. Thus $\angle ADJ = \angle JDE$ and so DJ is the angle bisector of $\angle ADE$. So J is the incentre of $\triangle ADE$, and it lies on $\mathcal{C}_1$.

(b) Let r_1 and r_2 denote the radiuses of the circles $\mathcal{C}_1$ and $\mathcal{C}_2$. Let O be the midpoint of the segment MN and R the midpoint of the segment DE.

$PQ \parallel MN$ as they are both perpendicular on AI. Thus $MNPQ$ is an isosceles trapezium with bases MN and PQ.

It is a rectangle if and only if the lengths of these segments are equal. We calculate them in terms of r_1 and r_2.

In $\triangle OIM$, we have $|OM| = |MI| \sin I$, while from the cosine formula in $\triangle IMJ$,

$$\cos I = \frac{2r_1^2 - r_2^2}{2r_1^2} = 1 - \frac{r_2^2}{2r_1^2}, \quad \text{thus} \qquad \sin I = \sqrt{1 - \cos^2 I} = \frac{r_2\sqrt{4r_1^2 - r_2^2}}{2r_1^2}.$$

We obtain $\quad |MN| = 2|OM| = \dfrac{r_2\sqrt{4r_1^2 - r_2^2}}{r_1}.$

On the other hand, $\triangle JPQ$ and $\triangle IDE$ are similar as they have parallel sides. This implies $|PQ|/|DE| = r_2/r_1$ and so $|PQ| = \frac{r_2}{r_1}|DE|$. On the other hand, $|DE|/2 = |ER| = |QE|$ as both ER and EQ are tangent to $\mathcal{C}_2$. From the trapezoid $JQEI$ with right angles at Q and E we get

$$|EQ|^2 = r_1^2 - (r_1 - r_2)^2 = 2r_1r_2 - r_2^2.$$

Thus $|DE| = 2\sqrt{2r_1r_2 - r_2^2}$ and so $|PQ| = \dfrac{2r_2}{r_1}\sqrt{2r_1r_2 - r_2^2}.$

From the above,

$$\begin{aligned} |MN| = |PQ| &\iff \sqrt{4r_1^2 - r_2^2} = 2\sqrt{2r_1r_2 - r_2^2} \\ &\iff (2r_1 - r_2)(2r_1 + r_2) = 4r_2(2r_1 - r_2) \\ &\iff 2r_1 = 3r_2. \end{aligned}$$

(By choice of incircles, $r_2 < r_1$ so $2r_1 - r_2 \neq 0$.)

5. In the mathematical talent show called "The X^2-factor", contestants are scored by a panel of 8 judges. Each judge awards a score of 0 ('fail'), X ('pass'), or X^2 ('pass with distinction'). Three of the contestants were Ann, Barbara and David. Ann was awarded the same score as Barbara by exactly 4 of the judges. David declares that he obtained different scores to Ann from at least 4 judges, and also that he obtained different scores to Barbara from at least 4 judges.

   In how many ways could scores have been allocated to David, assuming he is telling the truth?

   Solution (Proposed by Mark Flanagan)

   Represent each "score sheet" by a 8-digit ternary string with digits from $\{0, 1, 2\}$. Without loss of generality we may assume that Ann's score sheet reads 00001111, and that Barbara's score sheet reads 00002222. The total number of possible score sheets is 3^8. We will count the number of score sheets which could *not* have been allocated to David: call this number N.

   Denote by S_A the set of 8-digit ternary strings which differ from Ann's score in at most 3 places, and by S_B the set of 8-digit ternary strings which differ from Barbara's score in at most 3 places. Then

$$|S_A| = |S_B| = \binom{8}{0}2^0 + \binom{8}{1}2^1 + \binom{8}{2}2^2 + \binom{8}{3}2^3 = 577 \ .$$

   Next we count $|S_A \cap S_B|$. The condition $s \in |S_A \cap S_B|$ reduces to two possible cases:

   - The first 4 digits of s are all zero, and the second 4 digits contain at least one 1 and at least one 2. There are $3^4 - (2^4 + 2^4 - 1) = 50$ of these, since 2^4 contain no 1s, 2^4 contain no 2s, and 1 contains no 1s and no 2s.
   - Three of the first 4 digits of s are zero, and the second 4 digits contains exactly two 1s and exactly two 2s. There are $4 \cdot 2 \cdot \binom{4}{2} = 48$ of these.

   Therefore $|S_A \cap S_B| = 50 + 48 = 98$, and so

$$N = |S_A \cup S_B| = |S_A| + |S_B| - |S_A \cap S_B| = 2(577) - 98 = 1056 \ ,$$

   leaving the number of possible score sheets for David as $3^8 - N = 5505$.

   Solution 2:

   Represent each score sheet by a 8-digit ternary string with digits from $\{0, 1, 2\}$. Without loss of generality we may assume that Ann's score sheet reads 00001111, and that David's score sheet reads 00002222. Note that the number of differences between David's and Ann's scores in the first 4 digits is equal to the number of differences between David's and Barbara's scores in the first 4 digits.

   Next, note that the number of 4-digit ternary strings containing r nonzero elements is given by $a(r) = \binom{4}{r}2^r$. Also, denote by $b(r)$ the number of 4-digit ternary strings which differ from 1111 and 2222 in at least r digits, for $r = 0, 1, 2, 3, 4$. Then

- $b(0) = 3^4$ (all strings are allowed)
- $b(1) = 3^4 - 2$ (since all strings except 1111 and 2222 are allowed)
- $b(2) = 3^4 - 2^4 - 2$ (since disallowed strings are those which differ from 1111 and 2222 in at most one place)
- $b(3) = 1 + \binom{4}{3} \cdot 2 + \binom{4}{2} \cdot 2 = 21$ (partitioning the count on the number of zeros in the string)
- $b(4) = 1$ (only the string 0000 is allowed).

The total number of score sheets for David is then (partitioning the count according to the number of nonzero elements in the first 4 digits)

$$\begin{aligned}
N &= a(4)b(0) + a(3)b(1) + a(2)b(2) + a(1)b(3) + a(0)b(4) \\
&= \binom{4}{4} 2^4 \cdot 3^4 + \binom{4}{3} 2^3 \cdot (3^4 - 2) + \binom{4}{2} 2^2 \cdot (3^4 - 2^4 - 2) \\
&+ \binom{4}{1} 2^1 \cdot 21 + \binom{4}{0} 2^0 \cdot 1 \\
&= 3^4 \left(\binom{4}{4} 2^4 + \binom{4}{3} 2^3 + \binom{4}{2} 2^2 \right) - \binom{4}{3} 2^4 - \binom{4}{2} 2^6 - \binom{4}{2} 2^3 + 8 \cdot 21 + 1 \\
&= 3^4 \left(3^4 - 9 \right) - 2^6 - 3 \cdot 2^7 - 3 \cdot 2^4 + 169 \\
&= 3^8 - (3^6 + 2^6 + 2 \cdot 2^3 \cdot 3^3) + 169 \\
&= 3^8 - (3^3 + 2^3)^2 + 13^2 \\
&= 5505 \,,
\end{aligned}$$

i.e., the number of score sheets possible for David is 5505.