TWENTY THIRD IRISH MATHEMATICAL OLYMPIAD

Saturday, 24 April 2010

Second Paper

Problems and Solutions

6. There are 14 boys in a class. Each boy is asked how many other boys in the class have his first name, and how many have his last name. It turns out that each number from 0 to 6 occurs among the answers.

   Prove that there are two boys in the class with the same first name and the same last name.

   Solution (Proposed by Mark Flanagan)

   Consider groups of students with the same first name – these groups partition the 14 students. Also consider groups of students with the same last name – these groups also partition the 14 students. Each student belongs to two groups, and by assumption there are groups of size $1, 2, 3, 4, 5, 6$ and 7; but these numbers add up to $1 + 2 + 3 + 4 + 5 + 6 + 7 = 7(4) = 28$, so there is one group of each size from 1 to 7 and no other groups.

   Suppose without loss of generality the group of 7 is a group of students with the same first name. There are at most 6 groups by last name, so by the Pigeonhole Principle, two students in the group of 7 must also have the same last name.

7. For each odd integer $p \geq 3$ find the number of real roots of the polynomial

$$f_p(x) = (x-1)(x-2)\cdots(x-p+1)+1 \; .$$

Solution (Proposed by Bernd Kreussler)

We first look at the cases $p = 3, 5$, then deal with $p \geq 7$. The polynomial $f_3(x) = (x-1)(x-2)+1 = x^2-3x+3 = \left(x-\frac{3}{2}\right)^2 + \frac{3}{4} > 0$ has no real root. If $p = 5$, we obtain

$$\begin{aligned} f_5(x) &= (x-1)(x-2)(x-3)(x-4)+1 \\ &= (x-1)(x-4)\cdot(x-2)(x-3)+1 \\ &= (x^2-5x+4)(x^2-5x+6)+1 \\ &= \left(x^2-5x+5\right)^2 \; . \end{aligned}$$

Because the discriminant of x^2-5x+5 is $5^2-4\times 5 > 0$, we see that $f_5(x)$ has two real (double) roots.

If $p \geq 7$, we proceed as follows. First note that the degree of $f_p(x)$ is equal to $p-1$, hence this polynomial can have at most $p-1$ real roots. We obviously have $f_p(1) = f_p(2) = \ldots = f_p(p-1) = 1$. We shall show below that $f\left(2k-\frac{1}{2}\right) < 0$ for $k = 1, 2, 3, \ldots, \frac{p-1}{2}$. This implies the existence of two real roots between $2k-1$ and $2k$. Therefore, we conclude that $f_p(x)$ has exactly $p-1$ real roots, if $p \geq 7$ is an odd integer.

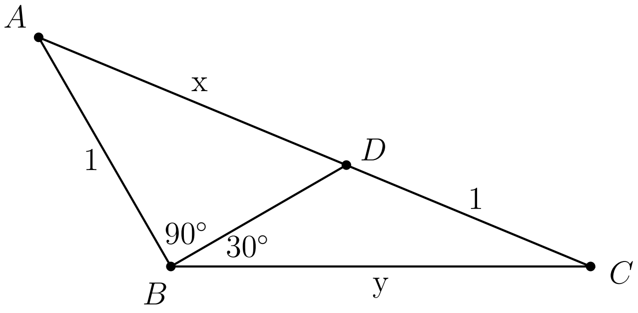
For $k = 1, 2, 3, \ldots, \frac{p-1}{2}$ we have

$$f_p\left(2k-\frac{1}{2}\right) = \frac{4k-3}{2}\cdot\ldots\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\left(-\frac{2p-4k-1}{2}\right)+1$$

The product has $p-1 \geq 6$ factors. Only $\pm\frac{1}{2}$ are of modulus less than 1. The number of negative factors is odd (equal to $p-2k$), so the product is negative. The modulus of the product is always at least $\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2} = \frac{105}{32} > 1$ except when $p = 7$ and $k = 2$, where it is equal to $\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2} = \frac{225}{64} > 1$. This shows that $f\left(2k-\frac{1}{2}\right) < 0$ for $k = 1, 2, 3, \ldots, \frac{p-1}{2}$.

8. In the triangle ABC we have $|AB| = 1$ and $\angle ABC = 120°$. The perpendicular line to AB at B meets AC at D such that $|DC| = 1$. Find the length of AD.

Solution (Proposed by Jim Leahy)



Let $x = |AD|$ and $y = |BC|$. From $\triangle ADB$ we obtain $\sin(\angle ADB) = \frac{1}{x}$ and from $\triangle BDC$ we get $\frac{\sin(\angle BDC)}{y} = \frac{\sin(30°)}{1} = \frac{1}{2}$. Since $\sin(\angle ADB) = \sin(\angle BDC)$, this gives $\frac{1}{x} = \frac{y}{2}$, hence $y = \frac{2}{x}$. The cosine theorem for $\triangle ABC$ gives

$$-\frac{1}{2} = \cos(\angle ABC) = \frac{1 + y^2 - (1+x)^2}{2y} = \frac{1 + \left(\frac{2}{x}\right)^2 - (1+x)^2}{\frac{4}{x}} .$$

Simplifying this, we obtain $x^4 + 2x^3 - 2x - 4 = 0$. Rewriting this polynomial as $x(x^3 - 2) + 2(x^3 - 2) = (x^3 - 2)(x + 2)$, we see that $x = \sqrt[3]{2}$.

9. Let $n \geq 3$ be an integer and $a_1, a_2, \ldots, a_n$ be a finite sequence of positive integers, such that, for $k = 2, 3, \ldots, n$

$$n(a_k + 1) - (n-1)a_{k-1} = 1 \; .$$

Prove that a_n is not divisible by $(n-1)^2$.

Solution (Proposed by Bernd Kreussler)

The given recursion can be rewritten as $a_{k-1} = \frac{n}{n-1}a_k + 1$ for $2 \leq k \leq n$. Let us write, for simplicity, $q = \frac{n}{n-1}$, so that we have $a_{k-1} = qa_k + 1$. We obtain

$$\begin{aligned} a_{n-1} &= qa_n + 1 \\ a_{n-2} &= qa_{n-1} + 1 = q^2 a_n + q + 1 \\ a_{n-3} &= qa_{n-2} + 1 = q^3 a_n + q^2 + q + 1 \end{aligned}$$

and we guess the general formula $a_{n-k} = q^k a_n + q^{k-1} + q^{k-2} + \ldots + 1$, which is easily shown by induction. In particular, with $k = n-1$, we obtain

$$a_1 = q^{n-1}a_n + q^{n-2} + q^{n-3} + \ldots + 1 = q^{n-1}a_n + \frac{q^{n-1} - 1}{q - 1} \; .$$

We now use $q = \frac{n}{n-1}$, from which we get $\frac{1}{q-1} = n - 1$ and

$$a_1 = q^{n-1}a_n + (n-1)(q^{n-1} - 1) = \frac{n^{n-1}a_n + (n-1)n^{n-1}}{(n-1)^{n-1}} - (n-1) \; .$$

Because a_1 is an integer, this implies that $n^{n-1}a_n + (n-1)n^{n-1}$ has to be divisible by $(n-1)^{n-1}$. Because $\gcd(n-1, n) = 1$, this is equivalent to $a_n + (n-1)$ being divisible by $(n-1)^{n-1}$. Hence, there exists an integer K, such that $a_n = -(n-1) + K(n-1)^{n-1} = (n-1)\left(K(n-1)^{n-2} - 1\right)$. As $n \geq 3$, $K(n-1)^{n-2} - 1 \equiv -1 \not\equiv 0 \mod (n-1)$, hence a_n is not divisible by $(n-1)^2$.

Remark: By induction, we can prove that $a_{k+1} = K(n-1)^k n^{n-1-k} - (n-1)$. This shows that $a_2, a_3, \ldots, a_n$ are divisible by $(n-1)$, but $a_3, \ldots, a_n$ are not divisible by $(n-1)^2$ and a_2 is divisible by $(n-1)^2$ iff $K \equiv 1 \mod (n-1)$.

10. Suppose a, b, c are the side lengths of a triangle ABC. Show that

$$x = \sqrt{a(b+c-a)}, \quad y = \sqrt{b(c+a-b)}, \quad z = \sqrt{c(a+b-c)}$$

are the side lengths of an acute-angled triangle XYZ, with the same area as ABC, but with a smaller perimeter, unless ABC is equilateral.

Solution (Proposed by Finbarr Holland)

Let $2s = a+b+c$, and denote by Δ the area of ABC. Then $x = \sqrt{2a(s-a)}$, etc., and $\Delta^2 = s(s-a)(s-b)(s-c)$. To show that XYZ is acute, because of the cosine theorem, we must show x^2, y^2, z^2 satisfy the triangle inequality. But, e.g.,

$$x^2 < y^2+z^2, \iff a(s-a) < b(s-b)+c(s-c), \iff b^2+c^2-a^2 < s(b+c-a),$$

i.e.,

$$2(b^2+c^2-a^2) < (b+c)^2-a^2, \iff (b-c)^2 < a^2, \iff (b-c-a)(b-c+a) < 0,$$

which is true. Also,

$$\begin{aligned}\cos(X) &= \frac{y^2+z^2-x^2}{2yz} = \frac{(c+a-b)(a+b-c)}{4\sqrt{bc(s-b)(s-c)}} \\ &= \frac{4(s-b)(s-c)}{4\sqrt{bc(s-b)(s-c)}} = \sqrt{\frac{(s-b)(s-c)}{bc}} = \sin\left(\frac{A}{2}\right)\end{aligned}$$

Hence,

$$\begin{aligned}yz\sin(X) &= 2\sqrt{bc(s-b)(s-c)}\cos\left(\frac{A}{2}\right) \\ &= 2\sqrt{bc(s-b)(s-c)}\sqrt{\frac{s(s-a)}{bc}} \\ &= 2\sqrt{s(s-a)(s-b)(s-c)} = 2\Delta\ .\end{aligned}$$

Thus ABC and XYZ have the same area. However,

$$x \le \frac{a+b+c-a}{2} = \frac{b+c}{2}\ ,$$

with equality iff $2a = b+c$, whence

$$x+y+z \le \frac{(b+c)+(c+a)+(a+b)}{2} = a+b+c\ ,$$

with equality iff $2a = b+c, 2b = c+a, 2c = a+b$, i.e., $a = b = c$. Thus, the perimeter of XYZ is not greater than the perimeter of ABC.