TWENTY THIRD IRISH MATHEMATICAL OLYMPIAD

Saturday, 24 April 2010

First Paper

Problems and Solutions

1. Find the least k for which the number 2010 can be expressed as the sum of the squares of k integers.

   Solution (Proposed by Rachel Quinlan)

   2010 is a multiple of 3 but not of 9. The sum of two squares can be a multiple of 3 only if the two squares involved are both multiples of 3 – which means they must be multiples of 9 as well. So 2010 is not the sum of two squares. 2010 can be written in various ways as the sum of three squares. For completeness, all possible such representations are listed below.

$$\begin{aligned} 2010 &= 1^2 + 28^2 + 35^2 \\ &= 4^2 + 25^2 + 37^2 \\ &= 5^2 + 7^2 + 44^2 \\ &= 5^2 + 31^2 + 32^2 \\ &= 7^2 + 19^2 + 40^2 \\ &= 11^2 + 17^2 + 40^2 \\ &= 16^2 + 23^2 + 35^2 \\ &= 19^2 + 25^2 + 32^2 \; . \end{aligned}$$

2. Let ABC be a triangle and let P denote the midpoint of the side BC. Suppose that there exist two points M and N interior to the sides AB and AC respectively, such that

$$|AD| = |DM| = 2|DN|,$$

where D is the intersection point of the lines MN and AP. Show that $|AC| = |BC|$.

Solution (Proposed by Anca Mustata)

Step I: Reduce to the case when M coincides with B. The following provides two versions of this step:

Version 1: The parallel line from B to MN intersects AP at a point E and AC at a point Q. From the similar triangles AMD and ABE it follows that

$$\frac{|MD|}{|BE|} = \frac{|AD|}{|AE|}$$

and from the similar triangles ADN and AEQ it follows that

$$\frac{|ND|}{|QE|} = \frac{|AD|}{|AE|}.$$

Thus $|AD| = |DM| = 2|DN|$ is equivalent to $|AE| = |EB| = 2|EQ|$.

Version 2: Let $B'C'$ be a line parallel to BC, with $B' = M$ and C' on AC. Let P' be the intersection point of $B'C'$ with AP. From the similar triangles $AB'P'$ and ABP it follows that

$$\frac{|B'P'|}{|BP|} = \frac{|AP'|}{|AP|}$$

and from the similar triangles $AC'P'$ and ACP it follows that

$$\frac{|C'P'|}{|CP|} = \frac{|AP'|}{|AP|}.$$

Thus P is the midpoint of BC if and only if P' is the midpoint of $B'C'$. Also, since the triangles $AB'C'$ and ABC are similar, then $|AC| = |BC|$ is equivalent to $|A'C'| = |B'C'|$.

Step II: With the notations from Step I, Version 1, prove that Q is the midpoint of AC and E is the centroid of the triangle ABC. The following provides two versions of this step:

Version 1: By Menelaus' theorem for the triangle BQC with secant line AP it follows that Q is the midpoint of the side AC. The triangles CPQ and CBA are similar since

$$\frac{|CQ|}{|AC|} = \frac{|CP|}{|BC|} = \frac{1}{2}$$

and so PQ is parallel to AB. Thus the triangles AEB and QEP are similar which implies

$$\frac{|AE|}{|EP|} = \frac{|BE|}{|EQ|} = 2 \ .$$

Version 2: Let Q' denote the midpoint of AC, and let E' be the intersection of BQ' with AP. We will prove $Q = Q'$ using proof by contradiction. Assume they are different. The triangles CPQ' and CBA are similar since

$$\frac{|CQ'|}{|AC|} = \frac{|CP|}{|BC|} = \frac{1}{2}$$

and so $\frac{|PQ'|}{|AB|} = \frac{1}{2}$ and also PQ' is parallel to AB. Thus the triangles $AE'B$ and $PE'Q'$ are similar which implies

$$\frac{|AB|}{|PQ'|} = 2 = \frac{|BE'|}{|E'Q'|} = \frac{|BE|}{|EQ|} \ .$$

But this would imply that EE' is parallel to QQ' which is a contradiction, since they meet in A.

Step III: Then the triangles AEQ and BEP are congruent, so

$$\frac{1}{2}|BC| = |BP| = |AQ| = \frac{1}{2}|AC| \ .$$

3. Suppose x, y, z are positive numbers such that $x+y+z=1$. Prove that

(a) $xy+yz+zx \geq 9xyz$;

(b) $xy+yz+zx < \frac{1}{4}+3xyz$.

Solution (Proposed by Finbarr Holland)

(a) Through two uses of the arithmetic mean - geometric mean inequality,

$$xyz = \sqrt[3]{(xy)(yz)(zx)} \cdot \sqrt[3]{xyz} \leq \frac{xy+yz+zx}{3} \cdot \frac{x+y+z}{3} = \frac{xy+yz+zx}{9}$$

with equality iff $x=y=z=1/3$. Hence,

$$xy+yz+zx \geq 9xyz\ ,$$

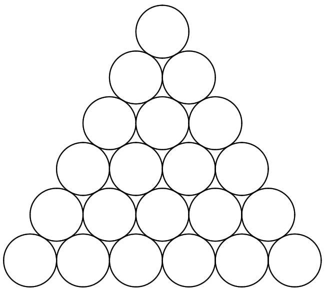
i.e. inequality (a) holds.

(b) Next, at least one of x, y, z must be at least $1/3$. Without loss of generality, suppose $z \geq 1/3$. Then

$$\begin{aligned} xy+yz+zx-3xyz &= xy(1-3z)+z(x+y) \\ &= xy(1-3z)+z(1-z) \\ &\leq z(1-z) \\ &= \frac{1}{4}-\left(z-\frac{1}{2}\right)^2 \\ &\leq \frac{1}{4}\ , \end{aligned}$$

and there is equality iff $xy=0$ and $z=1/2$. Since $xyz>0$, the inequality is strict. Thus inequality (b) holds.

4. The country of Harpland has three types of coin: *green*, *white* and *orange*. The unit of currency in Harpland is the shilling. Any coin is worth a positive integer number of shillings, but coins of the same colour may be worth different amounts. A set of coins is stacked in the form of an equilateral triangle of side n coins, as shown below for the case of $n = 6$.



The stacking has the following properties:

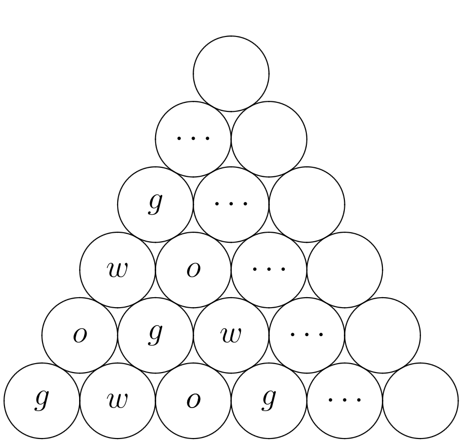
(a) no coin touches another coin of the same colour;

(b) the total worth, in shillings, of the coins lying on any line parallel to one of the sides of the triangle is divisible by three.

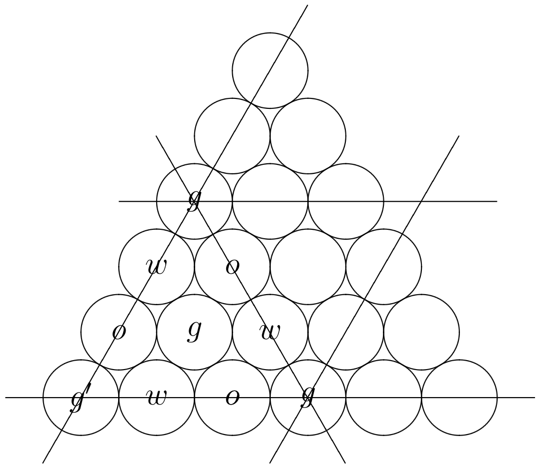
Prove that the total worth in shillings of the *green* coins in the triangle is divisible by three.

Solution (Proposed by Mark Flanagan)

Without loss of generality the coins form the pattern shown in the diagram below.



Let G, W and O denote the total worth in shillings of the green, white and orange coins in the triangle, respectively. The problem reduces to showing that G, W and O are *all* divisible by three. Now fix some green coin g' in the triangle, and consider a subset of the lines parallel to the sides of the triangle, such that every third parallel line is included, and all lines passing through g' are included; this is illustrated below.



Since the total worth of the coins on each line is divisible by 3, so is the sum of these quantities. However these lines contain each white coin exactly once, each orange coin exactly once, and a subset $\mathcal{S}$ of the green coins exactly 3 times. We thus have

$$W + O + 3G' \equiv 0 \pmod 3$$

where G' denotes the total worth of the green coins lying in the set $\mathcal{S}$. Therefore

$$W + O \equiv 0 \pmod 3 \ . \tag{1}$$

But summing the worth of the coins in all lines parallel to one side of the triangle, we obtain

$$G + W + O \equiv 0 \pmod 3 \ .$$

Subtracting (1) from this equation yields

$$G \equiv 0 \pmod 3 \ ,$$

and application of the same reasoning proves that $W \equiv O \equiv 0 \pmod 3$.

5. Find all polynomials $f(x) = x^3 + bx^2 + cx + d$, where b, c, d are real numbers, such that $f(x^2 - 2) = -f(-x)f(x)$.

Solution (Proposed by John Murray)

Solution 1: The 'obvious' approach is to equate coefficients of powers of x in $f(x^2 - 2)$ and $-f(-x)f(x)$ to get:

$$b - 6 = 2c - b^2, \quad -2bd + c^2 = 12 - 4b + c, \quad -d^2 = -8 + 4b - 2c + d. \quad (2)$$

It is possible, albeit difficult, to solve these equations directly. For example, if $b = 0$, we get $c = -3$ and then $d^2 + d - 2 = 0$, whence $d = 1$ or -2. Both

$$x^3 - 3x + 1, \quad \text{and} \quad x^3 - 3x - 2$$

are solutions, as we show below. If on the contrary, $b \neq 0$, we see that:

$$c = (b^2 + b)/2 - 3, \quad d = (c^2 - c + 4b - 12)/(2b), \quad d^2 + d + 4b - 2c - 8 = 0.$$

On substituting for c and then for d, and cancelling x^2, we see that b is a root of the sextic:

$$x^6 + 4x^5 - 22x^4 - 40x^3 + 129x^2 + 36x - 108.$$

The roots of this are $-6, -3, -1, 1, 2$ and 3. These could be found by guessing! Each root gives one solution for $f(x)$.

Solution 2: Set $a := 2$ and let $\beta_1, \beta_2, \beta_3$ denote the roots of f. Then the hypothesis implies that

$$\prod_{i=1}^{3}(x - \sqrt{a + \beta_i})(x + \sqrt{a + \beta_i}) = \prod_{i=1}^{3}(x - \beta_i)(x + \beta_i) \quad (3)$$

We consider the various possibilities.

Assume first that $\sqrt{a + \beta_i} = \pm\beta_i$, for all $i = 1, 2, 3$. Then each β_i is a root of $x^2 - x - 2$. So $\beta_i = -1$ or 2. It can be checked that this gives four possible polynomials $f(x)$:

$$(x + 1)^3, \quad (x + 1)^2(x - 2), \quad (x + 1)(x - 2)^2, \quad (x - 2)^3. \quad (4)$$

Assume next that $\sqrt{a + \beta_1} = \pm\beta_1$ but $\sqrt{a + \beta_2} \neq \pm\beta_2$. Then $\sqrt{a + \beta_2} = \pm\beta_3$ and so $\sqrt{a + \beta_3} = \pm\beta_2$. Now $\beta_1 = -1$ or 2, as before. Also $\beta_2 = \beta_3^2 - 2$ and $\beta_3 = \beta_2^2 - 2$. So β_2, β_3 are roots of

$$(x^2 - 2)^2 - x - 2 = (x^2 - x - 2)(x^2 + x - 1)$$

and hence the two complex conjugate roots $(-1 \pm \sqrt{5})/2$ of $x^2 + x - 1$. In this way we obtain two additional possible polynomials $f(x)$:

$$(x + 1)(x^2 + x - 1), \quad (x - 2)(x^2 + x - 1). \quad (5)$$

Finally we consider the case that $\sqrt{a+\beta_i} \neq \pm\beta_i$, for $i = 1, 2, 3$. Then we may choose notation so that (with i considered mod 3):

$$\sqrt{a+\beta_{i+1}} = \pm\beta_i,$$

whence $\beta_{i+1} = \beta_i^2 - a$, for $i = 1, 2, 3$. Thus $f(x)$ has the three roots:

$$\beta_i, \quad \beta_i^2 - a, \quad (\beta_i^2 - a)^2 - a = \beta_i^4 - 2a\beta_i^2 + a^2 - a.$$

Now $-b$ is the sum of the roots of f. So β_i is a root of the quartic

$$g(x) := x^4 + (1-2a)x^2 + x + a^2 - 2a + b = 0. \tag{6}$$

As its roots are distinct, it follows that $f(x)$ divides $g(x)$. As $g(x)$ has zero x^3 term, its easy to see that the quotient g/f must be $x - b$. Thus

$$(x-b)(x^3 + bx^2 + cx + d) = x^4 + (1-2a)x^2 + x + a^2 - 2a + b. \tag{7}$$

Thus we get $c - b^2 = 1 - 2a$ and $d - bc = 1$, or

$$c = b^2 - 2a + 1, \quad \text{and} \quad d = bc + 1. \tag{8}$$

Combining this with the first equality in (2), we deduce that $2(b^2 - 2a + 1) = b^2 + b - 3a$. So $b^2 - b + (2-a) = 0$. As $a = 2$, we solve to get $b = 0$ or 1. Thus we get two additional possible polynomials $f(x)$:

$$x^3 - 3x + 1, \quad x^3 + x^2 - 2x - 1. \tag{9}$$

The set of all possible polynomials $f(x)$ is contained in (4), (5) and (9).