Group Project 1 for Project Maths, Strand 2, Higher Level, Leaving Certificate
Drawing Basic Details for the First Quadrant of the Face of a Twelve-hour Clock

by
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TASK. Draw a quarter circle of radius-length 10cm so as to be the first quadrant of the face of a 12-hour clock. Mark in the hours 1 and 2 in large font and placed outside, and the minutes 1,2,3, 4 and 5 in small font and placed inside.

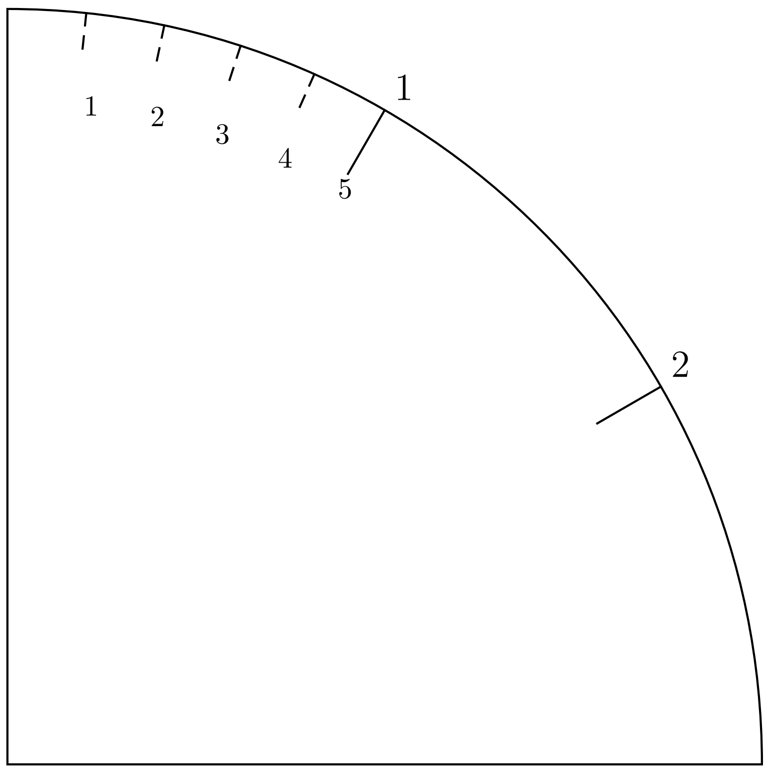

Figure 1.

This diagram was drawn using a software graphics package to show what is being sought but of course the task is <u>not</u> to use such a means.

1 Sizes of angles on a clock face

There are 360° in 12 hours so there are 30° for each hour. Similarly there are 360° in 60 minutes so there are 6° for each minute. It is well-known how to draw angles of size 30° and 60° and to calculate their sines and cosines. Thus our real work concerns angles of size 6°.

2 Products of lengths of intercepts of a line through a fixed point with a circle

For a fixed circle $\mathcal{C}(O;k)$ and fixed point $P \notin \mathcal{C}(O;k)$, let a variable line l through P meet $\mathcal{C}(O;k)$ at R and S. Then the product of distances $|PR||PS|$ is constant. When P is exterior to the circle,

$$|PR||PS| = |PT_1|^2,$$

where T_1 is the point of contact of a tangent from P to the circle.

Proof.

We first take P interior to the circle. Let M be the mid-point of R and S so that M is the foot of the perpendicular from O to RS. Then P is in either $[R, M]$ or $[M, S]$; we suppose that $P \in [R, M]$.

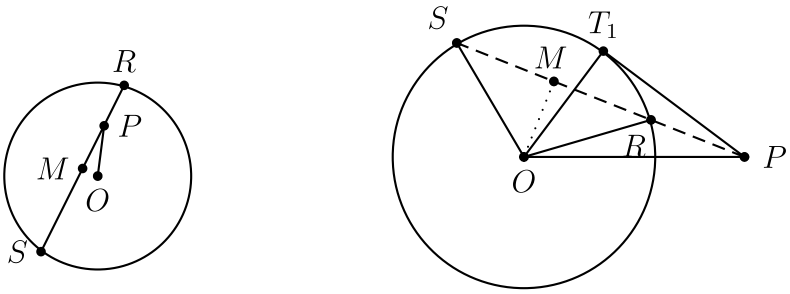

Figure 2.

Then

$$\begin{aligned} |PR||PS| &= (|MR| - |PM|)(|MS| + |PM|) = |MR|^2 - |PM|^2 \\ &= |MR|^2 - \left(|PO|^2 - |OM|^2\right) = \left(|MR|^2 + |OM|^2\right) - |PO|^2 \\ &= |OR|^2 - |PO|^2 = k^2 - |PO|^2, \end{aligned}$$

and this is fixed.

We continue with the case where P is exterior to the circle, and may suppose that $|PR| < |PS|$, as otherwise we can just interchange the points R and S. As P is outside the circle, it is outside the segment $[R, S]$ on the line RS. Then we have

$$\begin{aligned} |PR||PS| &= (|PM| - |MR|)(|PM| + |MS|) = |PM|^2 - |MR|^2 \\ &= \left(|PO|^2 - |OM|^2\right) - |MR|^2 = |PO|^2 - \left(|OM|^2 + |MR|^2\right) \\ &= |PO|^2 - |OR|^2 \\ &= |PO|^2 - |OT_1|^2 = |PT_1|^2. \end{aligned}$$

A converse of the second part of this result is also true, namely,: if for a point $T_3 \in \mathcal{C}(O; k)$ we have $|PR||PS| = |PT_3|^2$ for distinct points R and S on the circle, collinear with P, then PT_3 is a tangent to the circle at T_3. For then we must have $|PT_3| = |PT_1| = |PT_2|$ and so the circle on $[O, P]$ as diameter and the original circle would have three points in common, which cannot occur. Thus T_3 must be either T_1 or T_2.

3 Division of a line-segment in a special ratio

Given an arbitrary segment $[A, B]$ *to construct a point* $D \in [A, B]$ *such that* $|AB||BD| = |AD|^2$.

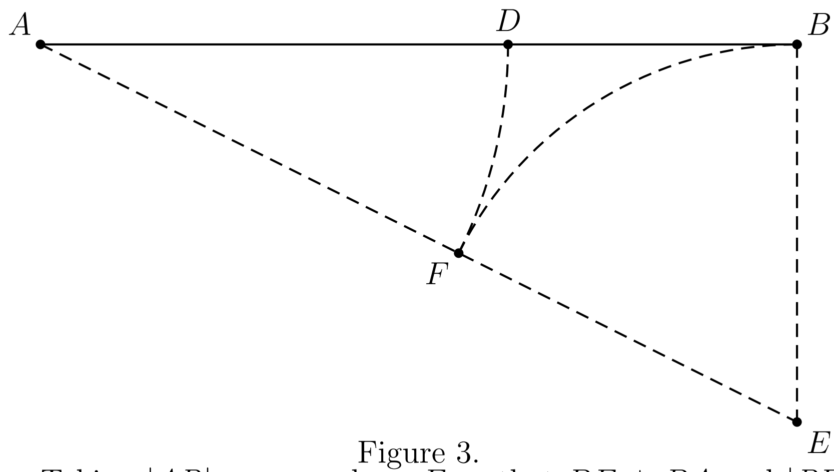

Figure 3.

Taking $|AB| = a$, we draw E so that $BE \perp BA$ and $|BE| = \frac{1}{2}a$. Draw the segment $[A, E]$ and find the point F on it such that $|EF| = |EB|$. Then

$|AE| = \frac{\sqrt{5}}{2}a$ and $|AF| = |AE| - |EF| = \frac{\sqrt{5}-1}{2}a$. Now find D on $[A, B]$ so that $|AD| = |AF|$. Then $|AD| = \frac{\sqrt{5}-1}{2}a$ and so $|DB| = a - \frac{\sqrt{5}-1}{2}a = \frac{3-\sqrt{5}}{2}a$. Now we note that $|AD|^2 = \left(\frac{\sqrt{5}-1}{2}a\right)^2 = \frac{3-\sqrt{5}}{2}a^2 = |AB||DB|$, as required.

4 Equilateral triangles

Take an arbitrary segment $[B, C]$ of length d. Let D be the mid-point of $[B, C]$ and erect a perpendicular $[D, A]$ to BC where $|DA| = \frac{\sqrt{3}}{2}d$. Draw the segments $[A, B]$ and $[A, C]$. Then for the triangle ABD, by Pythagoras' theorem we have that $|AB|^2 = |BD|^2 + |AD|^2 = \frac{1}{4}d^2 + \frac{3}{4}d^2 = d^2 = |BC|^2$. Similarly $|AC|^2 = |BC|^2$. Thus the triangle ABC has three sides of equal lengths and is called *an equilateral triangle*.

It follows that the circle $\mathcal{C}_1$ with centre B and radius length $|BC|$ and the circle $\mathcal{C}_2$ with centre C and radius length $|CB|$ both pass through the point A and so intersect there. Knowing this we can use the traditional technique for erecting an equilateral triangle on $[B, C]$ by finding A as a point of intersection of these two circles.

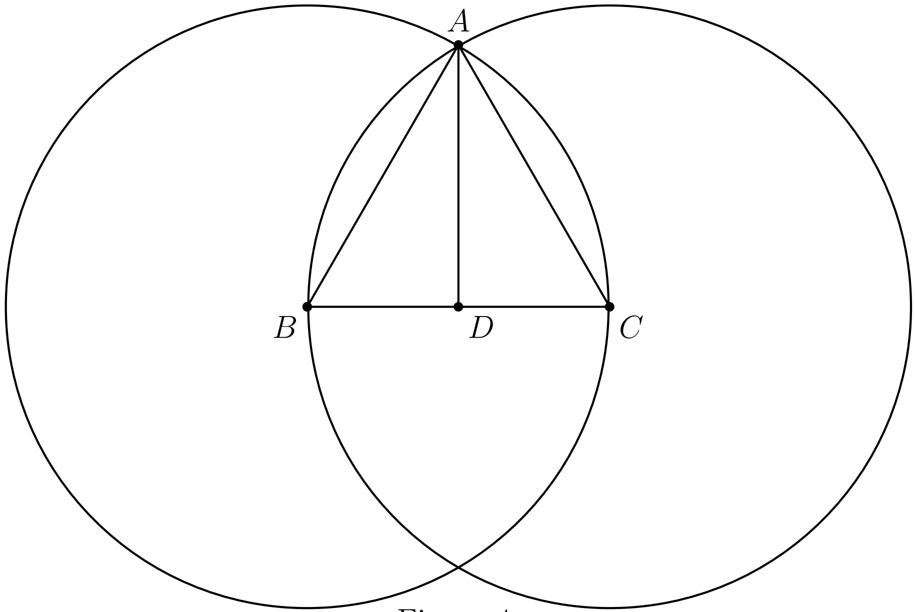

Figure 4.

5 To transfer a distance from one place to another

Given any segment $[A, B]$ and any point P other than A and B, we wish to construct a point X such that $|PX| = |AB|$. First we locate a point C such that APC is an equilateral triangle, and draw in the sides.

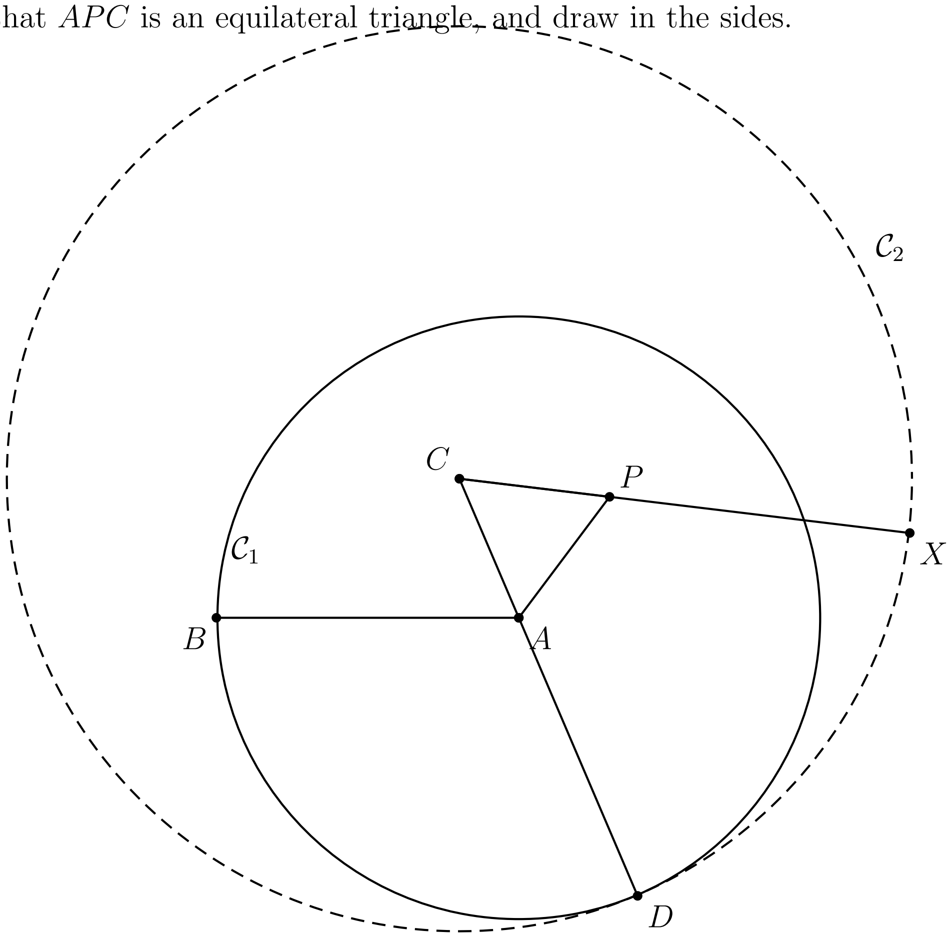

Figure 5.

Let $\mathcal{C}_1$ be the circle with centre and passing through B. Let $[C, A$ meet this circle at the point D. With C as centre draw the circle passing through D and denote it by $\mathcal{C}_2$. let $[C, P$ meet this second circle in the point X. Then

$$|CX| = |CD|,$$
$$|CP| = |CA|$$

so on subtracting the second line here from the first we obtain

$$|PX| = |CD| - |CA| = |AD| = |AB|.$$

This construction works if we take $P \notin AP$, or $P \in [A, B]$ but not an end point, or if P is on the line AB but outside the segment $[A, B]$ and it is this final case that we use in our application.

6 To construct a triangle ABC which has angles of measurement $36°, 72°, 72°$

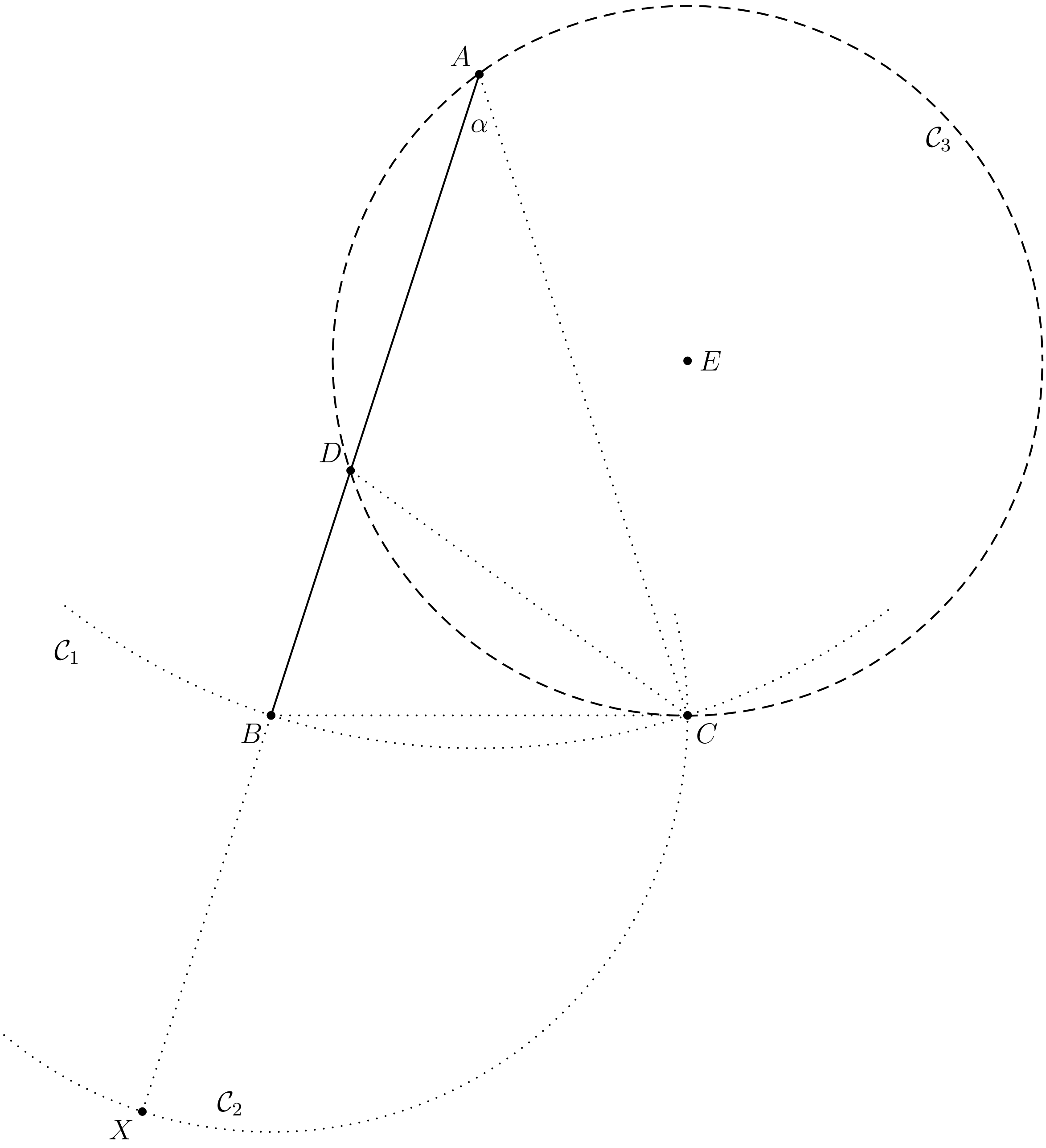

Figure 6.

Draw $[A, B]$ with D in it as in §3 . With A as centre and $|AB|$ as radius length draw an arc of a circle which we denote by $\mathcal{C}_1$. Choose the point X in $[A,\ B$ with $B \in [A, X]$ and $|BX| = |AD|$. Then we have extended $[A, B]$ to $[A, X]$ with the extension $[B, X]$ having the same length as $[A, D]$.

Draw an arc of a circle with centre B and radius length $|B, X|$, which we denote by $\mathcal{C}_2$. Let C be the point of intersection of these arcs of $\mathcal{C}_1$ and $\mathcal{C}_2$. Draw the segments $[A, C]$, $[B, C]$ and $[C, D]$. Then ABC is an isosceles triangle. Let us denote the angle size $|\angle BAC|$ by α.

Also draw the circumcircle $\mathcal{C}_3$ of the points A, D and C; its centre E is the point of intersection of any two of the perpendicular bisectors of $[A, C]$, $[A, D]$ and $[D, C]$. Now $|BC|^2 = |BA||BD|$ and so by the converse in §2 the line BC is a tangent to the circumcircle $\mathcal{C}_3$ at C.

Now $\angle BDC$ is an exterior angle for the triangle DAC and so $|\angle BDC| = |\angle DAC| + |\angle DCA|$. Moreover $|\angle BCD| = |\angle DAC| = \alpha$, as an angle between a tangent and a chord is equal to an angle in the alternate segment.

But $[C,\ D$ lies between $[C,\ B$ and $[C,\ A$ and so $|\angle BCA| = |\angle BCD| + |\angle DCA|$. Hence $|\angle BDC| = |\angle BCA| = |\angle CBA|$. It follows that the triangle CBD is isosceles with $|CD| = |CB|$. It follows that the triangle DAC is also isosceles, with $|DC| = |DA|$ and hence But then $|\angle DCA| = |\angle DAC| = \alpha$. It follows that $|\angle BDC| = 2\alpha$ and so $|\angle ABC| = |\angle ACB| = 2\alpha$.

Now the sum of the measures of the three angles of the triangle ABC is $180°$ so that $5\alpha = 180$ and so $\alpha = 36$.

On bisecting an angle of an equilateral triangle we obtain an angle of $30°$ and on drawing this inside and angle of $30°$ with the same vertex and one arm the same, by subtraction we have an angle of magnitude $6°$.

If we now consider the foot of the perpendicular from A to BC we can see from the measurements in §3 that $\cos 2\alpha = \frac{\sqrt{5}-1}{4}$ and hence $\sin 2\alpha = \frac{\sqrt{10+2\sqrt{5}}}{4}$, from which we obtain that

$$\cos\alpha = \frac{\sqrt{5}+1}{4},\ \sin\alpha = \frac{\sqrt{10-2\sqrt{5}}}{4}.$$

This gives that

$$\cos 36° = \frac{\sqrt{5}+1}{4},\ \sin 36° = \frac{\sqrt{10-2\sqrt{5}}}{4}.$$

7 Use of trigonometry

From

$$\sin 30^\circ = \tfrac{1}{2}, \quad \cos 30^\circ = \tfrac{\sqrt{3}}{2},$$

we can deduce that

$$\sin 6^\circ = \sin(36^\circ - 30^\circ) = \frac{\sqrt{3}\sqrt{10-2\sqrt{5}} - \sqrt{5} - 1}{8},$$

and

$$\cos 6^\circ = \cos(36^\circ - 30^\circ) = \frac{\sqrt{3}(\sqrt{5}+1) + \sqrt{10-2\sqrt{5}}}{8}.$$

Recalling that for 1 minute on the clock-face we require an angle of 84° in standard position, we can find the cosine or sine of this and plot the end of the upper arm on the circle.

We can verify our basic result by means of trigonometry as follows. As $\alpha = 36$ then $5\alpha = 180, 3\alpha = 180 - 2\alpha$ so that

$$\begin{aligned} \cos 3\alpha = \cos(180 - 2\alpha) &= -\cos 2\alpha, \\ \cos 3\alpha + \cos 2\alpha &= 0. \end{aligned}$$

We apply this in two ways. First we note that

$$\begin{aligned} 2\cos\frac{3\alpha+2\alpha}{2}\cos\frac{3\alpha-2\alpha}{2} &= 0, \\ \cos\frac{5\alpha}{2}\cos\frac{\alpha}{2} &= 0, \\ \cos\frac{5\alpha}{2} = 0 \quad \text{or} \quad \cos\frac{\alpha}{2} &= 0. \end{aligned}$$

From this we deduce that either

$$\begin{aligned} \frac{5\alpha}{2} = 90 + 180n \quad \text{or} \quad \frac{\alpha}{2} &= 90 + 180n, \\ \alpha = 36 + 72n, \quad \text{or} \quad \alpha &= 180 + 360n, \end{aligned}$$

for some integer n.

For a second application we note that

$$4\cos^3\alpha - 3\cos\alpha + 2\cos^2\alpha - 1 = 0.$$

If we write $\cos\alpha = x$ then we have that

$$4x^3 + 2x^2 - 3x - 1 = 0.$$

We can verify by long division that $x - \frac{\sqrt{5}+1}{4}$ is a factor of the expression on the left-hand side here, with the quotient being $4x^2 + (\sqrt{5}+3)x + \sqrt{5} - 1$.

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