

## SUMS AND GEOMETRY

**Main idea:** we will compute formulas for sums of consecutive numbers, or powers of consecutive numbers, and other related sums, by expressing them geometrically and trying to visualize them. Students usually learn of such sum formulas together with the method of induction. For example, prove:

$$1^1 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

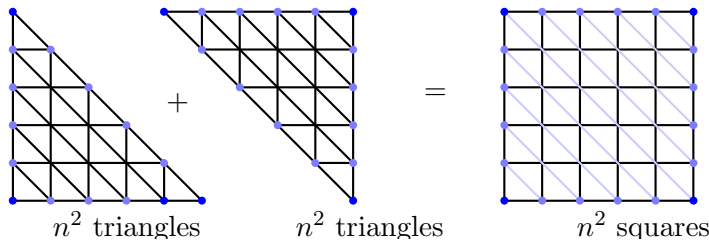
This example of induction is unfortunately quite a blind method, wherein you are given a rule and are asked to prove it, without gaining an inkling of where the formula came from. On the contrary, a genuine induction method should illuminate the path as well as the final goal, offering strategies for rule discovery and generalization. For this, visualization should come to most of us as natural and intuitive. It does have limitations given by the fact that we live in a 3 dimensional world, but it turns out that if we understand the rules governing 3 dimensions well enough we can generalize and even get some intuition of how these objects may look like in higher dimensions!

Here  $n > 0$  is a whole number.

- (1) **A triangle is divided uniformly into smaller triangles, by cutting each side into  $n$  pieces. How many such small triangles are there? (Can you find the exact number when  $n = 100$ ?)**

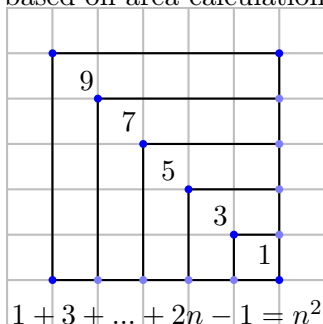
Working idea: Start with cases  $n = 2$ , then  $n = 3$ , etc. Counting the triangles row by row, we see that we need to add the first  $n$  consecutive odd numbers. On the other hand, when doing the sums in each case, one quickly comes to guess that the sum is  $n^2$ .

Geometric explanation: Students can construct these triangles using e.g. matchsticks and modeling clay, or Kinex kits, or Zome kits. If two teams each work at one large triangles, at some point you can bring the two teams together, and they might come up with the idea of putting together the two large triangles to form a large square. There are as many small triangles in a large one as there are small squares in a large one, that is  $n^2$ .



So we proved:  $1 + 3 + 5 + \dots + (2n - 1) = n^2$ .

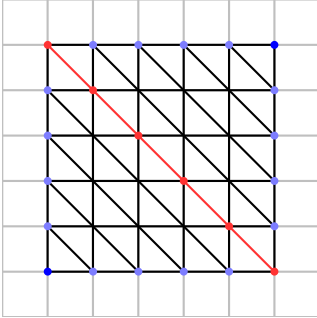
Another geometric solution for the same formula (in this case due to 10 year old Anna) is based on area calculation:



- (2) A triangle is divided uniformly into smaller triangles, by cutting each side into  $n - 1$  pieces. How many dots (vertices) are there in the picture? (can you find the exact number when  $n = 100$ ?)

Working ideas: This becomes  $1 + 2 + \dots + n = \frac{n(n+1)}{2}$ .

Method I: There are many ways to compute this, but since this is clearly related to the first question, one idea will be to form a large square out of two large triangles as before. But in this case the red dots on the diagonal are part of both triangles:

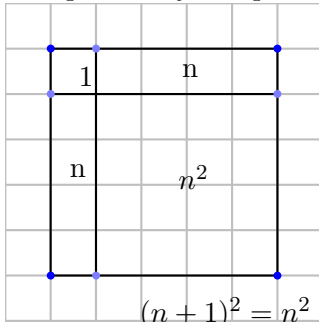


So this is a nice opportunity to apply the inclusion-exclusion principle:

$$\begin{aligned} \{ \text{dots in the first triangle} \} \cup \{ \text{dots in the second triangle} \} &= \{ \text{dots in the square} \} \\ |\{ \text{dots in the first large triangle} \}| + |\{ \text{dots in the second large triangle} \}| - n(\text{dots on the diagonal}) &= n^2, \\ 2|\{ \text{dots in a large triangle} \}| &= n(n+1). \end{aligned}$$

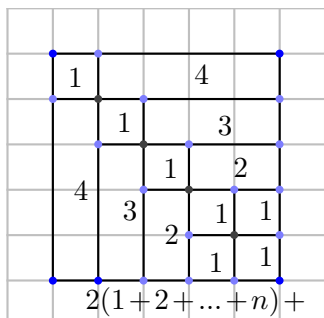
Method 2: Put together two large triangles without overlapping, to form a large rectangle of sides  $n$  and  $n + 1$ , then count the dots and divide by 2.

Method 3: Just like Anna used square areas to prove the first formula, the second one can be proven by completing the square:



$$\begin{aligned} (n+1)^2 &= n^2 + 2n + 1, \\ n^2 &= (n-1)^2 + 2(n-1) + 1, \\ (n-1)^2 &= (n-2)^2 + 2(n-2) + 1, \\ &\dots\dots\dots \\ 1^2 &= 0^2 + 2 \cdot 0 + 1. \end{aligned}$$

Add up, cancel some terms to get  $(n+1)^2 = 2 \sum_{k=0}^n k^2 + (n+1)$ , (the last term obtained by adding the 1-s), so  $(n+1)n = 2 \sum_{k=0}^n k^2$ .

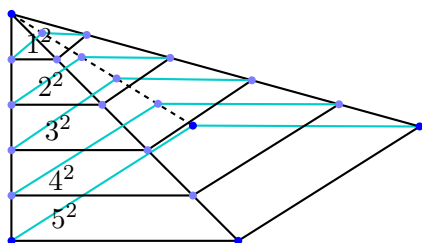


The algebra here looks awkward and more like a great overkill, but it has the advantages that it generalizes to finding sums of the  $k$ -th power of consecutive numbers for any  $k$ . All we'll have to do is complete the cube, then the hypercube...

- (3) A square of side  $n$  is divided uniformly into squares of side 1. How many squares can you count in the picture?

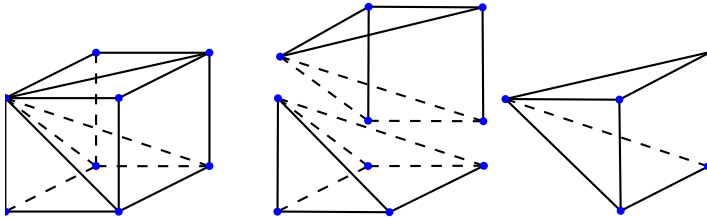
Working idea: We need to add  $n^2$  squares of size 1,  $(n - 1)^2$  squares of size 2,  $(n - 3)^2$  squares of size 3, etc.

Geometric solutions: Method 1. It's awkward to add  $n^2$  squares with  $(n - 1)^2$  squares with  $(n - 3)^2$  squares etc. when they're all on top of each other on the same plane. So we raise the problem in 3 dimensions. Construct a pyramid whose base is made of  $n^2$  squares (the small squares are not visible in these pictures, sorry. Best to use an actual 3d model). We'll count the dots (vertices of small squares) in the pyramid (including sides and edges).



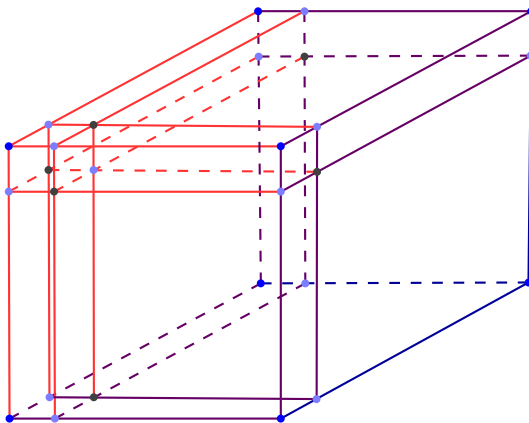
Putting together 3 such pyramids will yield a cube, but you'll need to glue along the three faces around the cube diagonal, so you get to apply the inclusion-exclusion principle again:

$$\begin{aligned}
 3|\{\text{dots in a pyramid}\}| - 3|\{\text{dots in a triangle}\}| + |\{\text{dots along diagonal}\}| &= n^3, \\
 3|\{\text{dots in a pyramid}\}| - 3\frac{n(n+1)}{2} + n &= n^3, \\
 |\{\text{dots in a pyramid}\}| &= \frac{n(n+1)(2n+1)}{6}.
 \end{aligned}$$



Method 2. Same idea, but no overlaps. I can't actually draw or imagine this without modeling it, but if you connect 6 pyramids like the one above so that the faces do not touch, but are separated by small edges, you'll get a box of sizes  $n$ ,  $(n + 1)$  and  $(2n + 1)$ . Unfortunately these methods are even harder to imagine in 4d and so generalize. But the next one works:

Method 3. Completing the cube, counting volume:



$$(n + 1)^3 = n^3 + 3n^2 + 3n + 1$$

Write this down for  $n, n - 1$ , etc. add

up, cancel terms just like in question (2)...

- (4) Can we generalize to compute the sums of any powers of consecutive numbers?

The algebra can be generalized to any dimensions by applying binomial coefficients:

$$(n + 1)^D = \sum_{k=0}^D \binom{D}{k} n^k.$$

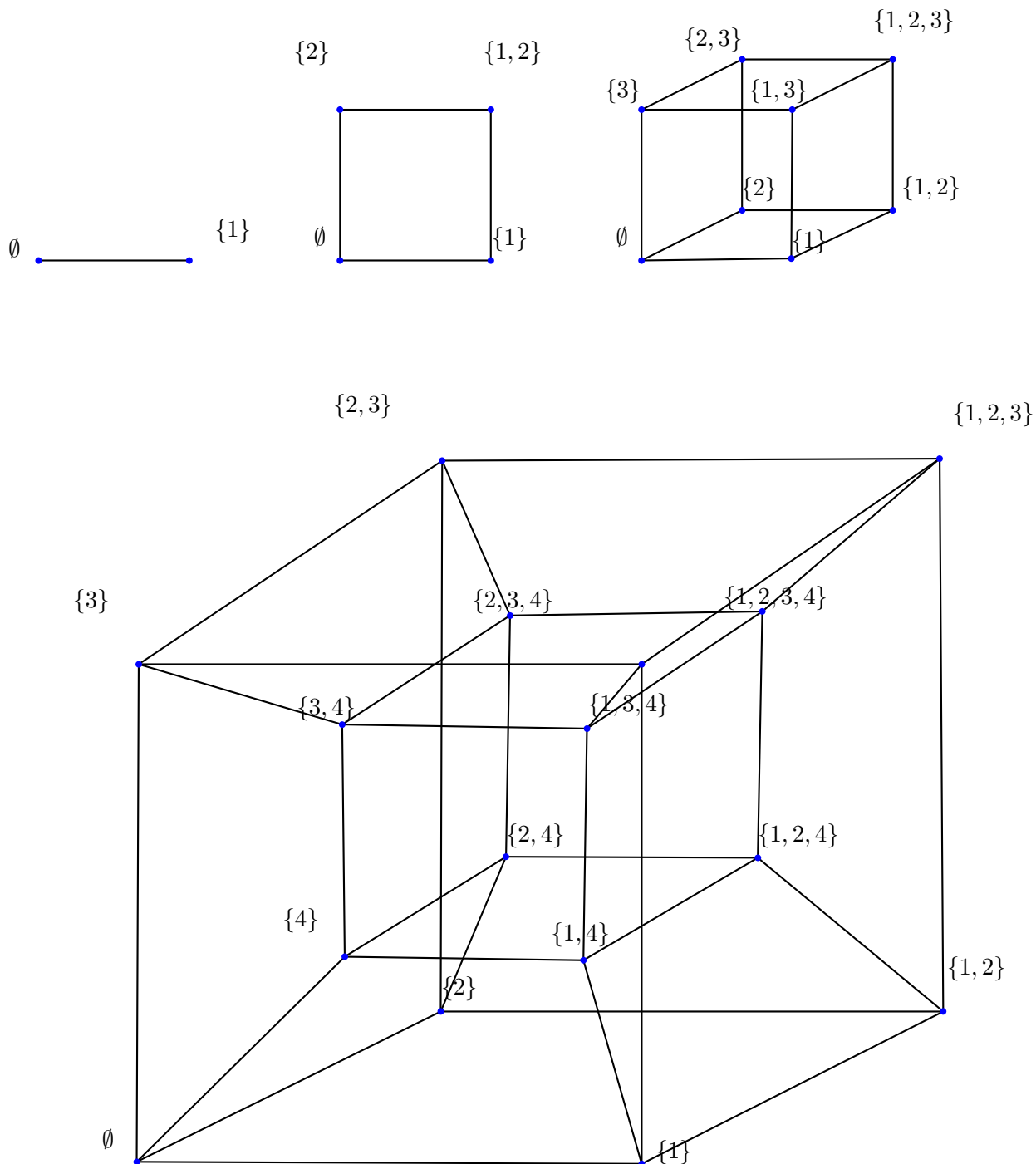
But can we have a geometric intuition of this formula? Can we imagine completing the  $D$ -dimensional cube? For this, we need a geometric equivalent for the binomial coefficients. In 3 dimensions, we can see them as follows:

- 3 thin long parallelepiped of volume  $n$  neighboring the small cube. They basically stretch along the 3 edges which meet at a fixed vertex.
- 3 thin, but long and wide parallelepiped of volume  $n$  neighboring the small cube. They

basically stretch along the 3 faces which meet at a fixed vertex.

So we can guess that the binomial coefficients count the number of edges, faces, etc in a  $D$ -dimensional cube. Now we need to imagine the  $D$ -dimensional cube.

Here is the road from the 1d to the 4d picture.



- (5) **How many vertices in a  $D$ -dimensional cube?**

By induction, there should be  $2^D$ .

- (6) **How many edges, faces, etc, meeting at a fixed point in a  $D$ -dimensional cube?**

We notice that  $2^D$  represents the number of choices of subsets of the set  $\{1, 2, \dots, D\}$ . We can draw a graph where each such subset is represented by a vertex, and set inclusions are edges, faces, etc. like in the picture above. In this way we represent a  $D$ -dimensional cube.

Application:

- (7) A cube of side  $n$  is divided uniformly into cubes of side 1. How many cubes are there in the picture?

Further related questions:

- (8) There are  $n + 1$  houses in a village and each house is connected with any other house by exactly one bridge. How many bridges are there in total?

(There are two ways to count this, and they lead to formula for a previous ex.)

- (9) A regular triangle of side  $n$  is divided uniformly into regular triangles of side 1. How many regular triangles of side 2 are there in the picture? (can you find the exact number when  $n = 100$ ?)

Hint: Counting the triangles with the same orientation as the original triangle, and then those upside down:  $1 + 2 + \dots + (n - 1) + 1 + 2 + \dots + (n - 3) = \frac{(n-1)n}{2} + \frac{(n-3)(n-2)}{2} = n^2 - 3n + 3$  for  $n > 1$ .

Alternatively, by the method applied for question (1), we complete the triangle to a parallelogram and thus double the number of side 2 triangles, but we have also added in those  $(n - 2)$  side 2 triangles which intersect the diagonal of the parallelogram and are thus neither included in the original triangle nor in its "mirror". We get  $(n - 1)^2$  parallelograms of side 2 in a parallelogram of side  $n$ . After subtracting  $(n - 2)$  we get  $(n - 1)^2 - (n - 2) = n^2 - 3n + 3$ .

- (10) A regular triangle of side  $n$  is divided uniformly into regular triangles of side 1. How many regular triangles are there in the picture?
- (11) Find a formula for  $1^2 - 2^2 + 3^2 - \dots + (2n + 1)^2 - (2n + 2)^2$  and calculate this when  $n = 100$ .
- (12) Find a short formula for  $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n + 1)$  (can you find the exact number when  $n = 100$ ?). Find a geometric proof for your formula.

Hint:  $n(n + 1)(n + 2)/3$ . Can compute based on formulas for  $1 + 2^2 + \dots + n^2$  and  $1 + 2 + \dots + n$ . Or, geometric picture in attached file.

- (13) Ann, Ben and Cian need to choose one present each from a total of  $(n + 2)$  presents labeled by numbers from 1 to  $n + 2$ . For example, one choice is: (Ann-1, Ben-2, Cian-3). How many such possible choices are there? Suppose that you know that  $(k + 2)$  is the largest label number among the three presents chosen. How many possible choices are there in this case? Can you use this to prove the formula in the previous exercise?

- (14) Find a short formula for  $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \dots + n(n + 1)(n + 2)$ .

Hint: like in previous ex. but with 4 people...

- (15) Can you generalize the above for  $1 \cdot 2 \cdot 3 \cdot \dots \cdot (k + 1) + \dots + n(n + 1)(n + 2) \dots (n + k)$ ?
- (16) Find a formula for  $1 + 2^4 + \dots + n^4$  and calculate the sum when  $n = 100$ .
- (17) Use induction to prove all formulas found in the previous exercises.
- (18) Consider the table:

$$\begin{array}{rcl} 1 & = & 1 \\ 3 + 5 & = & 8 \\ 7 + 9 + 11 & = & 27 \\ 13 + 15 + 17 + 19 & = & 64. \end{array}$$

Guess the general law suggested by these examples and prove it.

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