Sequences and recurrence relations.

1. For each of the following sequences, write the general term x_n and the sum

$$\sum_{k=0}^{n} x_k$$

in terms of n, x_0, (also x_1, if necessary) and the constants a, b.

a) Arithmetic progression: $x_{n+1} = x_n + a$ for all $n \in \mathbb{N}$.

b) Geometric progression: $x_{n+1} = ax_n$ for all $n \in \mathbb{N}$.

c) $x_{n+1} = ax_n + b$ for all $n \in \mathbb{N}$.

d) $x_{n+2} = ax_{n+1} + bx_n$ for all $n \in \mathbb{N}$.

2. (IMO1979) Let A and E be opposite vertices of an octagon. A frog starts at vertex A. From any vertex except E it jumps to one of the two adjacent vertices. When it reaches E it stops. Let a_n be the number of distinct paths of exactly n jumps ending at E. Prove that: $a_{2n-1} = 0$, $a_{2n} = \frac{(2+\sqrt{2})^{n-1}}{\sqrt{2}} - \frac{(2-\sqrt{2})^{n-1}}{\sqrt{2}}$.

3. (IMO1981) The function $f(x, y)$ satisfies: $f(0, y) = y + 1, f(x + 1, 0) = f(x, 1), f(x + 1, y + 1) = f(x, f(x + 1, y))$ for all non-negative integers x, y. Find $f(4, 1981)$.

4. (IMO1988) A function f is defined on the positive integers by: $f(1) = 1, f(3) = 3, f(2n) = f(n), f(4n + 1) = 2f(2n + 1) - f(n)$, and $f(4n + 3) = 3f(2n + 1) - 2f(n)$ for all positive integers n. Determine the number of positive integers $n \leq 1988$ for which $f(n) = n$.

5. (IMO2009) Suppose that $s_1, s_2, s_3, ...$ is a strictly increasing sequence of positive integers such that the subsequences

$$s_{s_1}, s_{s_2}, s_{s_3}, ... \text{ and } s_{s_1+1}, s_{s_2+1}, s_{s_3+1}, ...$$

are both arithmetic progressions. Prove that the sequence $s_1, s_2, s_3, ...$ is itself an arithmetic progressions.

6. A certain increasing function f satisfies the following relation when n is even:

$$f(n) = 4f(\frac{n}{2}) + n^2, f(1) = 1.$$

Find an expression for $f(2^k)$, for all integers $k \geq 0$.

7. We define a sequence of functions $P_k(X)$ by

$$P_1(X) = X^2 - 2 \text{ and } P_k(X) = P_1(P_{k-1}(X)).$$

For each n, find how many real numbers $x \geq \sqrt{2}$ satisfy the equation $P_n(x) = x$.

8. We define a sequence of functions $P_k(X)$ by

$$P_1(X) = \frac{1}{1-X} \text{ and } P_k(X) = P_1(P_{k-1}(X)).$$

Find $P_{2^{2016}}(2016)$.

9. We define a sequence of functions $P_k(X)$ by

$$P_1(X) = \frac{1}{2-X} \text{ and } P_k(X) = P_1(P_{k-1}(X)).$$

Find the real number x such that $P_{100}(x) = 2$.

10. A sequence of numbers c_n is defined by the following recurrence formula:

$$\sum_{k=0}^{m} \binom{m+p-k}{p} c_k = 1$$

for all non-negative integers m, where p is a fixed number.

Find the sum $\sum_{k=0}^{m} \binom{n}{k} c_k$.

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