Recurrence Relations

1. A fair coin is tossed 10 times. What is the probability that two heads do not occur consecutively.

2. Alison and Josh are standing at the top of a flight of 10 stairs. Alison can jump down one or two stairs a time (for example, she could get to the bottom by taking five jumps of two stairs at a time). In how many distinct ways can Alison get from the top of the stairs to the bottom? Josh can jump up to 10 stairs in one go. In how many distinct ways can Josh get from the top of the stairs to the bottom?

3. Consider sequences that consist entirely of A's and B's and that have the property that every run of consecutive A's has even length, and every run of consecutive B's has odd length. Examples of such sequences are AA, B, and $AABAA$, while $BBAB$ is not such a sequence. Derive a recurrence relation, and iterate to find how many such sequences have length 8?

4. Let A and E be opposite vertices of an octagon. A frog starts at vertex A. From any vertex except E it jumps to one of the two adjacent vertices. When it reaches E it stops. Let a_n be the number of distinct paths of exactly n jumps ending at E. Prove that:
$$a_{2n-1} = 0, \quad a_{2n} = \frac{(2+\sqrt{2})^{n-1} - (2-\sqrt{2})^{n-1}}{\sqrt{2}}$$

5. A mail carrier delivers mail to the nineteen houses on the east side of Elm Street. The carrier notices that no two adjacent houses ever get mail on the same day, but that there are never more than two houses in a row that get no mail on the same day.

   Derive the relevant recurrence relation to determine how many different patterns of mail delivery are possible, and calculate the necessary initial conditions. **Stop here, do not try to solve the characteristic equation by hand.**

   (Hint: Define houses either as 0 or 1 depending on whether or not they get delivered to. Then derive recurrence relations for certain well chosen binary sequences.)

6. In how many ways can you tile a 2×8 room with two types of tiles: a rectangle that is one unit wide and two units long, and an L-shaped tile covering three square units.

7. Suppose we are given a $2k \times 3$ chessboard, and $3k$ 2×1 rectangular tiles. The tiles are to be placed on the chessboard without overlap or gaps, such that the chessboard is completely covered. Prove that the number of distinct ways of doing so, N_k, satisfies
$$N_{2k} = 4N_{2k-2} - N_{2k-4}$$
   and that
$$N_{2k} = \left(1/2 + 1/2\sqrt{3}\right)\left(2+\sqrt{3}\right)^k + \left(1/2 - 1/2\sqrt{3}\right)\left(2-\sqrt{3}\right)^k$$

8. For positive integer n, prove that

$$\frac{(2+\sqrt{2})^n+(2-\sqrt{2})^n}{(2+\sqrt{2})^n-(2-\sqrt{2})^n}=\frac{1}{\sqrt{2}}\frac{2}{1+\frac{1}{1+\frac{2}{\cdots+\frac{\cdots}{1+2}}}}$$

where there are exactly $2n-2$ '+' signs on the RHS, and the numerators on the RHS follow the pattern 2,1,2,1, ...