UCC Mathematics Enrichment - Number Theory

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1. Find, with proof, all integer solutions of:

   (a) $a^3 + b^3 = 9$

   (b) $35x^3 + 66x^2y + 42xy^2 + 9y^3 = 9$

2. Find a pair of integers r, s such that $0 < s < 200$ and
$$\frac{45}{61} > \frac{r}{s} > \frac{59}{80}$$
   Also prove that there is exactly one such pair r, s.

3. Find all integers a, b, c for which
$$(x-a)(x-10)+1 = (x+b)(x+c)$$
   for all x.

4. Find a positive integer whose first digit is 1 and which has the property that, if this digit is transferred to the end of the number, the number is tripled.

5. Prove that the number
$$3^n + 2 \times 17^n$$
   where n is a non-negative integer, is never a perfect square.

6. What is the remainder when 2012^{2012} is divided by 11? What about
$$(\cdots(((2012^{2011})^{2010})^{2009})^{\cdots})^1?$$

7. Suppose that we have some positive integers (not necessarily distinct) whose sum is 100. How large can their product be?

8. Find all positive integers m and n, where n is odd, that satisfy
$$\frac{1}{m} + \frac{4}{n} = \frac{1}{12}$$

9. Let S be a set of rational numbers with the following properties:

   (a) $\frac{1}{2} \in S$;

   (b) If $x \in S$, then both $\frac{1}{x+1} \in S$ and $\frac{x}{x+1} \in S$.

   Prove that S contains all rational numbers in the interval $0 < x < 1$.

10. How many integer solutions are there to the equation $mn = 2016$?