Things one should know about triangles

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November 25, 2015

1 Introduction

The common or garden triangle is a fascinating geometric object of study for anybody with an enquiring mind, and down the ages it has provided lots of pleasure to a great many people. It's a source of an endless supply of challenging problems, which in turn suggest similar ones for other geometric plane figures. School-leavers with an aptitude for Mathematics should have a good understanding of the basic formulae associated with triangles, and be able to manipulate them with dexterity. They should also have a solid appreciation of their key properties. We will discuss some of the latter in these notes which attempt to present a coherent account of the somewhat off-the-cuff lecture I gave about them in class on November 21, 1915.

2 A review of some basic facts about triangles

2.1 The Sine and Cosine Rules

We follow standard notation for a triangle ABC, and review some facts that you probably know. $[ABC]$ stands for its area. This is given by "half the base by the perpendicular height". But this is only one way to find its area. For instance, applying this rule by taking each side as a "base" we have

$$[ABC] = \frac{ab\sin C}{2} = \frac{bc\sin A}{2} = \frac{ca\sin B}{2}.$$

So, knowing two sides and the included angle we can determine its area. These identities yield the Sine Rule:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{2[ABC]}{abc}.$$

Thus, knowing the lengths of two sides and the angle opposite one of these sides, we can determine the remaining side.

Exercise 1. *Deduce from the Sine Rule that* $b^2 \leq a^2 + b^2\cos^2 A$, *with equality iff* $\sin B = 1$.

Knowing all three side lengths we can determine the angles via the Cosine Rule:

$$\cos B = \frac{c^2 + a^2 - b^2}{2ca}, \ \cos C = \frac{a^2 + b^2 - c^2}{2ab}, \ \cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

keeping a cyclical notation.
What if we know two sides and the opposite angles? Can we find the remaining side length? Yes! Say a, b and $\angle A, \angle B$ are given. How can we find c? We can deduce it from the Cosine Rule as follows: we have

$$a \cos B = \frac{c^2 + a^2 - b^2}{2c}, \ b \cos A = \frac{b^2 + c^2 - a^2}{2c},$$

whence

$$a \cos B + b \cos A = \frac{2c^2}{2c} = c,$$

so that

$$c = a \cos B + b \cos A,$$

and similarly,

$$a = b \cos C + c \cos B, \ b = c \cos A + a \cos C.$$

Exercise 2. *Prove that*

$$a^2 \cos^2 B - b^2 \cos^2 A = a^2 - b^2.$$

Exercise 3. *Suppose a, b, c are three numbers, real or complex, and $abc \neq 0$. Solve the equations*

$$a = bz + cy, \ b = cx + az, \ c = ay + bx$$

for x, y, z.

What if we only know two sides, and one of the opposite angles?

Example 1. *Determine the area of the triangle two of whose side lengths are $3, 1$ and the cosine of one angle is $1/2$.*

Solution. This problem is not well-defined. Can we, for example, be sure that such a triangle can be constructed? Is there a unique triangle that satisfies the specifications? I leave you to answer the first of these questions.
Can we have $a = 3, c = 1$ and $\cos C = 1/2$? Try to use the Cosine Rule to find b, namely, solve

$$\frac{1}{2} = \frac{a^2 + b^- c^2}{2ab} \ \Leftrightarrow \ 3b = b^2 + 8, \ \Leftrightarrow \ b^2 - 3b + 8 = 0.$$

But this quadratic has no real solution. So, this configuration won't do.
Can we have $a = 1, c = 3$ and $\cos C = 1/2$? This time the Cosine Rule provides us with another quadratic equation for b: $b^2 - b - 8 = 0$, the positive solution of which is $b = (1 + \sqrt{33})/2$. Can we form a triangle with $a = 1, b = (1 + \sqrt{33})/2, c = 3$? See further on for a clue.

3 Other formulae for $[ABC]$

As well as the formulae mentioned already, another formula for the area is due to Heron. (Read about his other remarkable achievements in Google. He's credited with discovering the Reflection Principle, and inventing dispensing machines!)He proved that

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)},$$

where $2s = a+b+c$; s is the semi-perimeter of the triangle
Expanding the expression under the radical sign we obtain another formula for $[ABC]$ in terms of a, b, c, namely:

$$16[ABC]^2 = 2\sum a^2b^2 - \sum a^4.$$

Or, starting again from the Sine Rule, and using the Cosine Rule

$$\begin{aligned}
16[ABC]^2 &= 4a^2b^2 \sin^2 C \\
&= 4a^2b^2(1-\cos^2 C) \\
&= 4a^2b^2 - (a^2+b^2-c^2)^2 \\
&= 4a^2b^2 - (a^4+b^4+c^4-2a^2c^2-2b^2c^2+2a^2b^2) \\
&= 2a^2b^2+2b^2c^2+2c^2a^a-a^4-b^4-c^4 \\
&= 2\sum a^2b^2 - \sum a^4.
\end{aligned}$$

Corollary 1. *If a, b, c are side lengths of a triangle, then*

$$\sum a^4 < 2\sum a^2b^2.$$

NB. The inequality is strict: $<$ cannot be replaced by $\leq$.

Example 2. *No triangle exists with sides* $1, 2, 3$.

Proof. For

$1^4+2^4+3^4 = 1+16+81 = 98,\ 2(1^22^2+2^23^2+3^21^2) = 2(4+36+9) = 2(49) = 98$

Exercise 4. *Show that*

$$(x+y-z)^2 = x^2+y^2+z^2+2(xy-yz-zx).$$

Exercise 5. *Show that*

$$2(x+y+z)^2 = 6(xy+yz+zx)+(x-y)^2+(y-z)^2+(z-x)^2.$$

3.1 Another expression for $[ABC]$

Let

$$a = \frac{x+y}{2},\ b = \frac{y+z}{2},\ c = \frac{z+x}{2}.$$

Then

$$a+b+c = x+y+z,\ a+b-c = y,\ c+a-b = x,\ a+b-c = z,$$

i.e.,

$$2^4 s(s-a)(s-b)(s-c) = (x+y+z)xyz,$$

and $x, y, z > 0$.
Thus,

$$[ABC] = \frac{1}{4}\sqrt{(x+y+z)xyz},$$

which is more compact.
(The positivity of x, y, z comes from the triangle inequalities:

$$a < b+c,\ b < c+a,\ c < a+b$$

We'll say more about these in a later section.)

Exercise 6. *Suppose x, y, z are positive. Prove that*

$$3(x+y+z)xyz \leq (xy+yz+zx)^2.$$

4 Some inequalities

Mathematical inequalities are the life-blood of our subject, and arise practically in every branch of it. They come in all sorts of shapes and sizes. They can be discrete or continuous and can involve finitely or infinitely many variables. They can be sharp or crude. The latter type are often sufficient for the purpose in hand. Many of the most frequently occurring ones have been named.
In this section we'll first discuss the triangle inequalities which determine when three positive numbers are the side lengths of a triangle. Then we'll develop the inequality between the geometric mean and arithmetic mean of a finite set of positive numbers, and thereafter deal with simple cases of the isoperimetric inequality which aims to describe the largest planar region of a fixed type that can be enclosed by a curve of fixed finite length.

4.1 The triangle inequalities

A pair of arbitrary positive numbers can be regarded as the side lengths of a rectangle. By contrast, not every triple of positive numbers can be the side lengths of a

triangle. If a, b, c are positive, they are the side lengths of a triangle iff they satisfy the "triangle inequalities":

$$a < b + c, \;\; b < c + a, \;\; c < a + b.$$

These three inequalities can be expressed somewhat differently, first as a pair of inequalities,

$$|a - b| < c < a + b,$$

where $|x|$ stands for the absolute (numerical) value of a real number x defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$

Second as a single inequality:

$$2s = a + b + c > 2\max(a, b, c) \;\; \Leftrightarrow \;\; s > \max(a, b, c).$$

In all cases, the inequalities must be strictly satisfied, ie., we can't allow even one of them to be an equality. For instance, since, $1 + 2 + 3 = 6 = 2\max(1, 2, 3)$, the numbers $1, 2, 3$ are not the sides lengths of any triangle.

Example 3. *Given a pair of positive numbers x, y, and a real number $z \in (-1, 1)$, can a triangle be constructed two of whose side lengths are x, y and the cosine of one of its angles is z? If so, is it unique?*

Solution. A moment's thought should convince you that any given angle smaller than 180 degrees can be made so that its arms are any length. Call the angle A and its bounding arms b, c, with $b = x, c = y$. Join the extremities of the arms and we have a triangle of the desired kind.
An algebraic argument can be made by using the Cosine Rule. By setting $b = x, c = y$ and $\cos A = z$ we can use it to find a suitable a, and then confirm that the numbers a, b, c are the side lengths of a triangle. Indeed,

$$z = \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{x^2 + y^2 - a^2}{2xy}, \;\; \Leftrightarrow \;\; a^2 = x^2 - 2xyz + y^2 = (x - zy)^2 + y^2(1 - z^2).$$

So choose

$$a = \sqrt{(x - zy^2 + y^2(1 - z^2)}.$$

Then $a > 0$. In addition,

$$a^2 = b^2 - 2bcz + c^2 < b^2 + 2bc + c^2 = (b + c)^2 \;\; \Rightarrow \;\; a < b + c$$

But also,

$$a^2 > b^2 - 2bc + c^2 = (b - c)^2 \;\; \Leftrightarrow \;\; |b - c| < a.$$

Thus

$$|b - c| < a < b + c,$$

whence a, b, c are the side lengths of a triangle.
Can we make a triangle with $b = x, c = y$ and $\cos B = z$? I.e., can we solve the cosine formula

$$z = \cos B = \frac{c^2 + a^2 - b^2}{2ca} = \frac{y^2 + a^2 - x^2}{2ya} \Leftrightarrow a^2 - 2yza + y^2 - x^2 = 0,$$

for a? This is a quadratic equation in a. Its discriminant is equal to $4y^2z^2 - 4(y^2 - x^2) = 4(x^2 - (1 - z^2)y^2$ which is positive iff $x > \sqrt{1 - z^2}y$, which occurs if $x \geq y$. So, in the latter case we have $a = yz + \sqrt{x^2 - (1 - z^2)y^2}$. Do the three numbers a, b, c satisfy the triangle inequalities? If they do, we have another solution to our problem. Well,

$$a = yz + \sqrt{x^2 - (1 - z^2)y^2} \leq y + \sqrt{x^2 - (1 - z^2)y^2} < y + x = b + c$$

Trivially, $c = y < a < a + b$. But, by assumption, $c = y \leq x = b < b + a$. Thus, in this case the triangle inequalities hold. Of course, by switching the roles of b, c, we can also manufacture a triangle if $y \geq x$.
It's not always easy to select three positive numbers that are the side lengths of a triangle. But here's a helpful trick. Given any positive numbers x, y, z we can solve the equations

$$c + a - b = x, \ a + b - c = y, \ b + c - a = z,$$

for a, b, c, and be certain that they satisfy the triangle inequalities, and hence that they are the side lengths of some triangle. This transformation from x, y, z to a, b, c, and its inverse given by

$$a = \frac{x + y}{2}, \ b = \frac{y + z}{2}, \ c = \frac{z + x}{2},$$

are often very useful when handling inequalities involving the side lengths of a triangle.

4.2 The $AM - GM$ inequality

The simplest version of this says that the geometric mean of two positive numbers doesn't exceed their arithmetic mean, and that both means are equal iff the numbers are equal. In symbols, if $a, b > 0$, then

$$\sqrt{ab} \leq \frac{a + b}{2},$$

and the inequality is strict unless $a = b$.
This has a very simple proof. Because all the numbers involved are positive, the inequality is equivalent (?) to the statement that

$$4ab \leq (a + b)^2 \Leftrightarrow 0 \leq a^2 - 2ab + b^2 = (a - b)^2.$$

But the square of every real number is nonnegative; and is zero iff the number itself is zero. Hence, the result follows.

Exercise 7. *Prove that if a, b are real, then $2ab \leq a^2 + b^2$, with equality iff $a = b$.*

Exercise 8. *Prove that if a, b are real, then $2|ab| \leq a^2 + b^2$, with equality iff $|a| = |b|$.*

Here are a few simple consequences, which keep recurring in some form or other.

Example 4. *Suppose a, b, c are real numbers. Then*

$$ab + bc + ca \leq a^2 + b^2 + c^2,$$

with equality iff $a = b = c$.

Solution. Apply the first exercise of this section to each cross-product to get

$$2ab \leq a^2 + b^2,\ 2bc \leq b^2 + c^2,\ 2ca \leq c^2 + a^2.$$

Now add and divide by 2. Clearly, there is equality if $a = b = c$. Suppose, conversely that the equality holds. Then

$$0 = 2(a^2 + b^2 + c^2) - 2(ab + (bc + ca) = (a - b)^2 + (b - c)^2 + (c - a)^2,$$

a sum of three nonnegative real numbers whose sum is zero. Hence, each one is zero, etc..

Example 5. *Suppose a, b, c are real numbers. Then*

$$8(abc)^2 \leq (a^2 + b^2)(b^2 + c^2)(c^2 + a^2),$$

with equality iff $a = b = c$.

Solution. Apply the second exercise of this section to get

$$\begin{aligned} 8(abc)^2 &= 8a^2b^2c^2 \\ &= 8|a|^2|b|^2|c|^2 \\ &= (2|a||b|)(2|b||c|)(2|c||a|) \\ &\leq (a^2 + b^2)(b^2 + c^2)(c^2 + a^2). \end{aligned}$$

Exercise 9. *Prove that, if x, y, z are positive real numbers, then*

$$\sqrt{3(x + y + z)xyz} \leq (xy + yz + zx),$$

with equality iff $x = y = z$.

Exercise 10. *Prove that, if x, y, z are real numbers, then*

$$(x + y + z)^2 + x^2 + y^2 + z^2 \geq 4(xy + yz + zx),$$

with equality iff $x = y = z$.

Example 6. *If a, b, c are side lengths of a triangle, so are $\sqrt{a}, \sqrt{b}, \sqrt{c}$.*

Proof. Must show that $\sqrt{a} < \sqrt{b} + \sqrt{c}$, etc. The latter is equivalent to

$$a < (\sqrt{b} + \sqrt{c})^2 = b + c + 2\sqrt{bc}$$

which is true since $a < b + c < b + c + 2\sqrt{bc}$.
Note too that

$$\frac{\sqrt{a} + \sqrt{b} + \sqrt{c}}{3} \leq \sqrt{\frac{a+b+c}{3}},$$

with strict inequality unless $a = b = c$. For

$$\begin{aligned} 3(a+b+c) - (\sqrt{a} + \sqrt{b} + \sqrt{c})^2 &= 2(a+b+c) - 2\sum \sqrt{ab} \\ &= \sum (\sqrt{a} - \sqrt{b})^2 \\ &\geq 0, \end{aligned}$$

with equality iff $a = b = c$.

Exercise 11. *If a, b, c are side lengths of a triangle, so are $a/(1+a), b/(1+b), c/(1+c)$. How do the perimeters compare?*

Proposition 1. *Suppose x, y, z are positive and $a = \frac{x+y}{2}$, $b = \frac{y+z}{2}$, $c = \frac{z+x}{2}$ are the side lengths of a triangle ABC. Let $A'B'C'$ be the triangle whose side lengths are $\sqrt{a}, \sqrt{b}, \sqrt{c}$. Then*

$$[A'B'C'] = \frac{1}{4}\sqrt{xy + yz + zx}.$$

Proof. By a previous formula for the area of a triangle, derived in Section 3,

$$\begin{aligned} 16[A'B'C']^2 &= 2\sum (\sqrt{ab})^2 - \sum (\sqrt{a})^4 \\ &= 2\sum ab - \sum a^2 \\ &= \frac{1}{2}\sum (x+y)(y+z) - \frac{1}{4}\sum (x+y)^2 \\ &= \frac{1}{2}\sum (xy + yz + zx + y^2) - \frac{1}{4}\sum (x^2 + 2xy + y^2) \\ &= \frac{3}{2}\sum xy + \frac{1}{2}\sum x^2 - \frac{1}{2}\sum x^2 - \frac{1}{2}\sum xy \\ &= \sum xy. \end{aligned}$$

The stated result follows.

Proposition 2. *Suppose x, y, z are positive. Denote by $A'B'C'$ and $A''B''C''$ the triangles with side lengths $\sqrt{(x+y)/2}, \sqrt{(y+z)/2}, \sqrt{(z+x)/2}$ and $(\sqrt{x}+\sqrt{y})/2, (\sqrt{y}+\sqrt{z})/2, (\sqrt{z}+\sqrt{x})/2$, respectively. Then*

$$[A''B''C''] \leq [A'B'C'],$$

equality holding iff $x = y = z$.

Proof. The area of a triangle with side lengths a, b, c is given by either of the following expressions,

$$\frac{1}{4}\sqrt{2\sum a^2b^2 - \sum a^4},\ \sqrt{s(s-a)(s-b)(s-c)}.$$

If $a = (p+q)/2, b = (q+r)/2, c = (r+p)/2$, these are equal to $\frac{1}{4}\sqrt{(p+q+r)pqr}$. But also in the same notation, by Proposition 1, the area of the triangle with side lengths $\sqrt{a}, \sqrt{b}, \sqrt{c}$ is given by $\sum pq$
Applying this formula to $A'B'C'$ we get

$$4[A'B'C'] = \sqrt{xy + yz + zx},$$

and the previous one to $A''B'C''$, we get

$$4[A''B''C''] = \sqrt{(\sqrt{x} + \sqrt{y} + \sqrt{z})\sqrt{xyz}}.$$

It only remains to verify that

$$(\sqrt{x} + \sqrt{y} + \sqrt{z})\sqrt{xyz} \leq xy + yz + zx,$$

with equality only when $x = y = z$. Equivalently, with $u = \sqrt{xy}, v = \sqrt{yz}, w = \sqrt{zx}$, we require

$$uv + vw + wu \leq u^2 + v^2 + w^2,$$

which is a known result, with equality iff $u = v = w$. So, the proposition holds.

Exercise 12. *If a, b, c are side lengths of one triangle, show that $\sqrt[3]{a}, \sqrt[3]{b}, \sqrt[3]{c}$ are the side lengths of another.*

More generally, the $AM - GM$ inequality states that if $a_1, a_2, \ldots, a_n$ are n positive real numbers, then their geometric mean

$$G_n(a) \equiv G(a_1, a_2, \ldots, a_n) = \sqrt[n]{a_1 a_n \cdots a_n}$$

doesn't exceed their arithmetic mean

$$A_n(a) \equiv A(a_1, a_2, \ldots, a_n) = \frac{a_1 + a_2 + \cdots + a_n}{n},$$

and both means are equal iff $a_1 = a_2 = \cdots = a_n$.
The case $n = 4$ of this is deducible from the case $n = 2$. This is so because if a_1, a_2, a_3, a_4 are positive, then so are $b_1 = \sqrt{a_1a_2}, b_2 = \sqrt{a_3a_4}$, whence (?)

$$\begin{aligned}
G_4(a) &= G_2(b) \\
&= \sqrt{b_1b_2} \\
&\leq \frac{b_1 + b_2}{2} \\
&= \frac{\sqrt{a_1a_2} + \sqrt{a_3a_4}}{2} \\
&\leq \frac{\frac{a_1+a_2}{2} + \frac{a_3+a_4}{2}}{2} \\
&= A_4(a).
\end{aligned}$$

These inequalities are all strict unless $a_1 = a_2 = a_3 = a_4$.
We can build on this to handle the case when n is a power of two. How do we fill in the gaps between successive powers of 2? I'll provide a clue by treating the case $n = 3$.

Proposition 3. *If a, b, c are positive, then*

$$\sqrt[3]{abc} \leq \frac{a+b+c}{3}.$$

The equality sign holds iff $a = b = c$.

Proof. Let

$$d = \frac{a+b+c}{3},$$

and apply the case $n = 4$ to the four positive numbers a, b, c, d. This tells us that (?)

$$\sqrt[4]{abcd} \leq \frac{a+b+c+d}{4} = d \ \Leftrightarrow \ abcd \leq d^4 \ \Leftrightarrow \ abc \leq d^3,$$

as required with strict inequality unless $a = b = c = d$.

Exercise 13. *Establish that if x, y, z are arbitrary real or complex numbers, then*

$$x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx).$$

Deduce that, if a, b, c are positive, then

$$a^3 + b^3 + c^3 \geq 3abc,$$

with equality iff $a = b = c$.

4.3 Three instances of the isoperimetric inequality

Proposition 4. *Among all rectangles having the same perimeter, the square encloses the largest area.*

Proof. Given a rectangle with side lengths a, b, its area is ab, and its perimeter is $2(a+b)$. Suppose this is fixed, and equal to L, say. How big can ab be? In any event, since $2a < L$ an $2b < L$, we easily obtain the crude bound $L^2/4$ for ab. Thus, none of the prescribed rectangles has area greater than $L^2/4$. Can we refine this bound? Yes, by invoking the $AM - GM$ inequality for two numbers. According to this

$$\sqrt{ab} \leq \frac{a+b}{2} = \frac{L}{4},$$

and the equality holds iff $a = b = L/4$. Consequently, the square of side $L/4$ is the largest rectangle whose perimeter is L.
What's the story for triangles? Among all triangles having the same perimeter L, which one encloses the largest area? We're immediately confronted with a difficulty:

unlike the case of a rectangle, we have several formulae for the area of triangle. Which of these should we use, bearing in mind that the hypothesis involves the side lengths, viz., $a+b+c=L$? Will the Sine Rule work? According to this,

$$8[ABC]^3 = a^2b^2c^2 \sin A \sin B \sin C.$$

Since the sines don't exceed 1, we infer from the $AM-GM$ inequality for three numbers that

$$8[ABC]^3 \le (abc)^2 \;\Leftrightarrow [ABC] \le \frac{1}{2}(\sqrt[3]{abc})^2 \le \frac{1}{2}(\frac{a+b+c}{3})^2 = \frac{L^2}{18}.$$

Unfortunately, only one sine can equal 1, so we only obtain a crude estimate, namely that the largest triangle with perimeter L has area smaller than $L^2/18$. Can we perhaps refine this estimate by figuring out how large the product of the sines can be, and couple this with the $AM-GM$ inequality for a, b, c, and in this way arrive at a sharp result? In any event, it appears that the Sine Rule on its own doesn't suffice.
Would it be worthwhile narrowing the problem to a special class of triangles?

Proposition 5. *Among all right-angled triangles having the same perimeter, the isosceles one encloses the largest area.*

Proof. Suppose ABC has a right angle at C, so that $c^2 = a^2+b^2$, in which case its perimeter is given by $a+b+\sqrt{a^2+b^2}$, and its area is $\frac{1}{2}ab$. How big can this be if a, b are constrained by $a+b+\sqrt{a^2+b^2} = L$? Since $a+b \ge 2\sqrt{ab}$, and $a^2+b^2 \ge 2ab$, with equality in both iff $a=b$, we deduce that

$$L \ge 2\sqrt{ab} + \sqrt{2ab} = (2+\sqrt{2})\sqrt{ab},$$

whence

$$\frac{1}{2}ab \le \frac{L^2}{2(2+\sqrt{2})^2},$$

with equality iff

$$a = b = \frac{L}{2+\sqrt{2}}.$$

Since Heron's formula involves only a, b, c, it may be a more promising starting point. According to this

$$[ABC]^2 = s(s-a)(s-b)(s-c),$$

a product of four positive numbers, and the temptation is strong to apply the $AM-GM$ inequality for 4 numbers. Doing so, we see that

$$[ABC]^2 \le \Big(\frac{s+s-a+s-b+s-c}{4}\Big)^4 = \Big(\frac{4s-a-b-c}{4}\Big)^4 = \Big(\frac{L}{4}\Big)^4,$$

in which case

$$[ABC] \le L^2/16.$$

Thus the largest triangle has area at most $L^2/16$. But is this attainable? No, because equality can't hold in our application of the $AM - GM$ inequality. Fortunately we can recover the situation by applying the $AM - GM$ to the product $(s-a)(s-b)(s-c)$. This gives us

$$\sqrt[3]{(s-a)(s-b)(s-c)} \le \frac{s-a+s-b+s-c}{3} = \frac{L}{6},$$

with equality iff $a = b = c = L/3$. Thus,

$$[ABC]^2 \le \frac{L}{2}\left(\frac{L}{6}\right)^3 = \frac{L^4}{(16)(27)}, \Leftrightarrow [ABC] \le \frac{L^2}{12\sqrt{3}}.$$

In summary, we can assert:

Proposition 6. *Among all triangles with the same perimeter, the equilateral triangle encloses the largest area.*

We can avoid using the $AM - GM$ inequality to prove this as follows. Switch the data to x, y, z as described above. Then $x + y + z = L$, and

$$[ABC] = \frac{1}{4}\sqrt{(x+y+z)xyz}.$$

Now

$$3(x+y+z)xyz \le (xy+yz+zx)^2,$$

and

$$6(xy+yz+zx) + (x-y)^2 + (y-z)^2 + (z-x)^2 = 2(x+y+z)^2.$$

Hence

$$[ABC] = \frac{1}{4\sqrt{3}}\sqrt{3(x+y+z)xyz} \le \frac{1}{4\sqrt{3}}(xy+yz+zx) \le \frac{1}{12\sqrt{3}}(x+y+z)^2 = \frac{L^2}{12\sqrt{3}},$$

with equality iff $x = y = z = L/3$. Thus, among all triangles with perimeter L, the equilateral triangle with side $L/3$ encloses the largest area.
Our first attempt to establish the isoperimetric inequality for triangles through up a problem about the product

$$\sin A \sin B \sin C$$

which we didn't resolve. How big can it be? If ABC is equilateral, then $A = B = C = \pi/3$, in which case the value of the product is $3\sqrt{3}/8$. Hence, its maximum value is not smaller than this number. What is its maximum value? To find out it's convenient to establish the next proposition.

Proposition 7. *Suppose* $x_1, x_2, \ldots, x_n \in (0, \pi)$. *Then*

$$\sqrt[n]{\sin x_1 \sin x_2 \cdots \sin x_n} \leq \sin\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right),$$

with equality iff $x_1 = x_2 = \cdots = x_n$.

Proof. We'll only sketch the proof, which resembles that given for the $AM - GM$ inequality, for $n = 2, 3, 4$.
The case $n = 2$. So, let $0 < x, y < \pi$. We want to prove that

$$\sqrt{\sin x \sin y} \leq \sin(\frac{x + y}{2}),$$

with equality iff $x = y$. The inequality is equivalent to the next one

$$2 \sin x \sin y \leq 2 \sin^2(\frac{x + y}{2}) = 1 - \cos(x + y) = 1 - \cos x \cos y + \sin x \sin y.$$

I.e., to

$$\cos(x - y) = \cos x \cos y + \sin x \sin y \leq 1,$$

which is true, and the equality sign holds iff $x - y$ is a multiple of 2π. However, $-2\pi < x - y < 2\pi$, so the multiple has to be zero, so that $x = y$.
The case $n = 4$ follows from three applications of this. And this, in turn, implies the case $n = 3$, because if $x, y, z \in (0, \pi)$, set $w = (x + y + z)/3$. Then $0 < w < \pi$, and apply the case $n = 4$ to x, y, z, w etc.

Corollary 2. *If* A, B, C *are the angles in a triangle, then*

$$\sin A \sin B \sin C \leq \frac{3\sqrt{3}}{8},$$

with equality iff the triangle is equilateral.

Proof. Since the angle measures are less than π. we can say that

$$\sqrt[3]{\sin A \sin B \sin C} \leq \sin(\frac{A + B + C}{3}) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

Hence

$$\max \sin A \sin B \sin C = \frac{3\sqrt{3}}{8},$$

with equality iff the triangle is equilateral.

Corollary 3. *If* ABC *is any triangle, then*

$$[ABC] \leq \frac{\sqrt{3}\sqrt[3]{a^2b^2c^2}}{4}.$$

The equality holds iff ABC *is equilateral.*

Proof. By the Sine Rule,

$$8[ABC]^3 = a^2b^2c^2 \sin A \sin B \sin C,$$

which, by the previous corollary, doesn't exceed $a^2b^2c^2\, 3\sqrt{3}/8$. It follows that

$$[ABc]^3 \leq a^2b^2c^2(\frac{\sqrt{3}}{4})^3,$$

from which the stated result follows.
Here's another way to prove Proposition 7, which may have occurred to some readers. A straightforward application of the $AM - GM$ inequality tells us that

$$\sqrt[n]{\sin x_1 \sin x_2 \cdots \sin x_n} \leq \frac{\sin x_1 + \sin x_2 + \cdots + \sin x_n}{n},$$

with equality iff $\sin x_1 = \sin x_2 = \cdots = \sin x_n$. But this observation brings up a new problem: How big can the RHS be? Since the sin function is strictly concave on $(0, \pi)$—it's graph is like an inverted saucer—the latter is dominated by

$$\sin\Big(\frac{x_1 + x_2 + \cdots + x_n}{n}\Big),$$

and there is equality between the two iff the xs are equal. But this remark is only for the cognoscenti who know about convexity and such notions! It must be left for discussion another day.
By way of noting its usefulness, we have, for example,

Example 7. *In any triangle ABC*

$$\sin A + \sin B + \sin C \leq 3\sin(\frac{A+B+C}{3}) = 3\sin\frac{\pi}{3} = \frac{3\sqrt{3}}{2}.$$

Exercise 14. *Prove that in any triangle ABC*

$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$