UCC Mathematics Enrichment - Combinatorics

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1. $2m$ players are participating in a tennis tournament. Find the number of pairings for the first round.
2. Find R_n, the number of ways to place n undistinguishable rooks peacefully on an $n \times n$ chessboard. Find also H_n and Q_n, the number of those placings which are invariant with respect to a half-turn or quarter-turn, respectively.
3. Is it possible to label the edges of a cube by $1, 2, \ldots, 12$ so that, at each vertex, the labels of the edges leaving that vertex have the same sum?
4. Prove that $\left(\begin{array}{c} n \\ r \end{array} \right) \left(\begin{array}{c} r \\ k \end{array} \right) = \left(\begin{array}{c} n \\ k \end{array} \right) \left(\begin{array}{c} n-k \\ r-k \end{array} \right)$.
5. Consider all $2^n - 1$ nonempty subsets of the set $\{1, 2, \ldots, n\}$. For each subset, we find the product of the reciprocals of each of its elements. Find the sum of all these products.
6. How many strings (ordered sequences) of length 5 with values from $\{0, 1, \ldots, 9\}$ have (a) strictly increasing digits, (b) strictly increasing or decreasing digits, (c) increasing digits, (d) increasing or decreasing digits?
7. How many positive integer solutions to $a + b + c + d + e = 100$?
8. Find the number of subsets of $\{1, 2, \ldots, n\}$ that contain no consecutive elements of $\{1, 2, \ldots, n\}$.
9. Call a number "prime looking" if it is composite but not divisble by 2,3 or 5. The three smallest prime-looking numbers are 49, 77, and 91. There are 168 prime numbers less than 1000. How many prime-looking numbers are there less than 1000?
10. Among the 10 problems on this sheet, 3 are "easy", 3 are "challenging" and 3 are "difficult". Solve the problems in ascending order of difficulty. In how many different ways can you complete the problem set?