# Some Inequalities–variations on a common theme Lecture I, UL 2007

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## 1 Three Problems

**Problem 1.** *Assume* $a_i, b_i, c_i,\ i = 1, 2, 3$ *are real numbers. Prove that*

$$\sqrt{(a_1+b_1+c_1)^2+(a_2+b_2+c_2)^2+(a_3+b_3+c_3)^2} \le \sqrt{a_1^2+b_1^2+c_1^2}+\sqrt{a_2^2+b_2^2+c_2^2}+\sqrt{a_3^2+b_3^2+c_3^2}.$$

**Problem 2.** *Assume* $a_i, b_i, c_i,\ i = 1, 2, 3$ *are positive numbers. Prove that*

$$\sqrt[3]{a_1b_1c_1}+\sqrt[3]{a_2b_2c_2}+\sqrt[3]{a_3b_3c_3} \le \sqrt[3]{(a_1+a_2+a_3)(b_1+b_2+b_3)(c_1+c_2+c_3)},$$

*with equality iff*

$$\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}.$$

**Problem 3.** *Assume* $a_i, b_i, c_i,\ i = 1, 2, 3$ *are positive numbers. Prove that*

$$\frac{1}{\frac{1}{a_1}+\frac{1}{b_1}+\frac{1}{c_1}}+\frac{1}{\frac{1}{a_2}+\frac{1}{b_2}+\frac{1}{c_2}}+\frac{1}{\frac{1}{a_3}+\frac{1}{b_3}+\frac{1}{c_3}} \le \frac{1}{\frac{1}{a_1+a_2+a_3}+\frac{1}{b_1+b_2+b_3}+\frac{1}{c_1+c_2+c_3}},$$

*with equality iff*

$$\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}.$$

## 2 A basic inequality

**Theorem 1.** *Suppose* $a, b > 0$. *Then*

$$2\sqrt{ab} \le ta+\frac{b}{t},\ \forall t > 0,$$

*with equality iff*

$$t = \sqrt{\frac{b}{a}}.$$

*In other words,*

$$2\sqrt{ab} = \min\{ta + \frac{b}{t} : t > 0\} = \min\{sa + bt : s, t > 0, st = 1\}.$$

Proof.

$$ta + \frac{b}{t} - 2\sqrt{ab} = \frac{at^2 - 2t\sqrt{ab} + b}{t} = \frac{(t\sqrt{a} - \sqrt{b})^2}{t}.$$

We deduce some consequences.

**Theorem 2.** *If $a, b, c, d > 0$. Then*

$$\sqrt{ab} + \sqrt{cd} \leq \sqrt{(a+c)(b+d)},$$

*with equality iff*

$$\frac{a}{c} = \frac{b}{d}.$$

Proof. For all $t > 0$,

$$2[\sqrt{ab} + \sqrt{cd}] \leq at + \frac{b}{t} + ct + \frac{d}{t} = (a+c)t + \frac{(b+d)}{t},$$

with equality iff

$$t = \sqrt{\frac{b}{a}} = \sqrt{\frac{d}{c}} = \sqrt{\frac{b+d}{a+c}}.$$

Hence

$$2[\sqrt{ab} + \sqrt{cd}] \leq \min\{(a+c)t + \frac{(b+d)}{t} : t > 0\} = 2\sqrt{(a+c)(b+d)}.$$

We can build on this in two ways.

**Theorem 3.** *Suppose $a, b, c, d, e, f, g, h > 0$. Then*

$$\sqrt[4]{abcd} + \sqrt[4]{efgh} \leq \sqrt[4]{(a+e)(b+f)(c+g)(d+h)},$$

*with equality iff*

$$\frac{a}{e} = \frac{b}{f} = \frac{c}{g} = \frac{d}{h}.$$

Proof. Put $x = \sqrt{ab}, y = \sqrt{cd}$, $u = \sqrt{ef}, v = \sqrt{gh}$. Then

$$\begin{aligned}
\sqrt[4]{abcd} + \sqrt[4]{efgh} &= \sqrt{xy} + \sqrt{uv} \\
&\leq \sqrt{(x+u)(y+v)} \\
&= \sqrt{(\sqrt{ab} + \sqrt{ef})(\sqrt{cd} + \sqrt{gh})} \\
&\leq \sqrt{\sqrt{(a+e)(b+f)}\sqrt{c+g)(d+h)}} \\
&= \sqrt[4]{(a+e)(b+f)(c+g)(d+h)}.
\end{aligned}$$

We can now deduce the following statement.

**Corollary 1.** *If* $a, b, c, e, f, g > 0$, *then*

$$\sqrt[3]{abc} + \sqrt[3]{efg} \leq \sqrt[3]{(a+e)(b+f)(c+g)},$$

*with equality iff*

$$\frac{a}{e} = \frac{b}{f} = \frac{c}{g}.$$

Proof. Let $d = \sqrt[3]{abc}$, $h = \sqrt[3]{efg}$ and apply the theorem.

$$\begin{aligned} d + h &= \sqrt[4]{abcd} + \sqrt[4]{efgh} \\ &\leq \sqrt[4]{(a+e)(b+f)(c+g)(d+h)}, \end{aligned}$$

whence

$$\sqrt[3]{abc} + \sqrt[3]{efg} = d + h \leq \sqrt[3]{(a+e)(b+f)(c+g)}.$$

A solution of Problem 2 follows.

**Exercise 1.** *Prove the last statement.*

**Exercise 2.** *Suppose* $a_i, b_i > 0,\ i = 1, 2, \ldots, n$. *Prove that*

$$\sqrt[n]{a_1 a_2 \cdots a_n} + \sqrt[n]{b_1 b_2 \cdots b_n} \leq \sqrt[n]{(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)},$$

*with equality iff*

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}.$$

# 3 The Cauchy-Schwartz inequality

Theorem 2 is a special case of a disguised version of one form of perhaps the most useful inequality in all of Mathematics, namely the Cauchy-Schwartz inequality.

**Theorem 4 (Cauchy-Schwartz).** *If* $x_i, y_i > 0,\ i = 1, 2, \ldots, n$, *then*

$$\sum_{i=1}^{n} \sqrt{x_i y_i} \leq \sqrt{(\sum_{i=1}^{n} x_i)(\sum_{i=1}^{n} y_i)},$$

*with equality iff*

$$\frac{x_1}{y_1} = \frac{x_2}{y_2} = \cdots = \frac{x_n}{y_n}.$$

Proof. Invoke Theorem 2 and use induction. We can phrase this differently.

**Corollary 2.** *If* $a_i,\ i = 1, 2, \ldots, n$ *are real numbers, then*

$$\sqrt{\sum_{i=1}^{n} a_i^2} = \max\{|\sum_{i=1}^{n} a_i x_i| : \sum_{i=1}^{n} x_i^2 = 1\}.$$

Proof. For, if $\sum_{i=1}^n x_i^2 = 1$, then

$$\begin{aligned} |\sum_{i=1}^n a_i x_i| &\leq \sum_{i=1}^n |a_i|\,|x_i| \\ &= \sum_{i=1}^n \sqrt{(a_i)^2 (x_i)^2} \\ &\leq \sqrt{(\sum_{i=1}^n a_i^2)(\sum_{i=1}^n x_i^2)} \\ &= \sqrt{\sum_{i=1}^n a_i^2}, \end{aligned}$$

and there is equality when

$$x_i = \frac{a_i}{\sqrt{\sum_{i=1}^n a_i^2}}, \quad i = 1, 2, \ldots, n.$$

**Theorem 5.** *If $a, b, c, d$ are real numbers, then*

$$\sqrt{(a+b)^2 + (c+d)^2} \leq \sqrt{a^2 + c^2} + \sqrt{b^2 + d^2}.$$

Proof. Suppose $x^2 + y^2 = 1$. Then

$$|ax + cy| \leq \sqrt{a^2 + c^2}, \quad |bx + dy| \leq \sqrt{b^2 + d^2}.$$

Hence

$$|(a+b)x + (c+d)y| \leq |ax + cy| + |bx + dy| \leq \sqrt{a^2 + c^2} + \sqrt{b^2 + d^2}.$$

It follows that

$$\sqrt{(a+b)^2 + (c+d)^2} = \max\{|(a+b)x + (c+d)y| : x^2 + y^2 = 1\} \leq \sqrt{a^2 + c^2} + \sqrt{b^2 + d^2}.$$

This is a special case of Minkowski's inequality.

**Theorem 6 (Minkowski).** *Assume $a_i, b_i,\ i = 1, 2, 3$ are real numbers. Then*

$$\sqrt{\sum_{i=1}^n (a_i + b_i)^2} \leq \sqrt{\sum_{i=1}^n a_i^2} + \sqrt{\sum_{i=1}^n b_i^2}.$$

Proof. Suppose $\sum_{i=1}^n x_i^2 = 1$. Then

$$\begin{aligned} \sqrt{\sum_{i=1}^n a_i^2} + \sqrt{\sum_{i=1}^n b_i^2} &\geq |\sum_{i=1}^n a_i x_i| + |\sum_{i=1}^n b_i x_i| \\ &\geq |\sum_{i=1}^n (a_i + b_i) x_i|. \end{aligned}$$

Hence

$$\sqrt{\sum_{i=1}^{n}(a_i+b_i)^2} = \max\{|\sum_{i=1}^{n}(a_i+b_i)x_i| : \sum_{i=1}^{n} x_i^2 = 1\} \le \sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2}.$$

# 4 Discussion of Theorem 2

The product $ab$ represents the area of a rectangle $A$ with dimensions $a, b$, and $cd$ is the area of a rectangle $B$ with dimensions $c, d$. And $a+c, b+d$ are the dimensions of a rectangle $C$—the 'sum' of $A, B$. Thus

$$\sqrt{area(A)} + \sqrt{area(B)} \le \sqrt{area(C)},$$

with equality iff $A, B, C$ are scaled versions of each other.

**Exercise 3.** *Give a geometric interpretation of Problem 2.*

We can cast the previous statement in a different way. With each point (vector) $x = (x_1, x_2)$ is the first quadrant of $\mathbb{R}^2$, let

$$g_2(x) = \sqrt{x_1 x_2}.$$

Then, if $y = (y_1, y_2)$,

$$g_2(x) + g_2(y) \le g_2(x+y).$$

Thus $g_2$ is *super-additive* on $\mathbb{R}^2$.
Notice, too, that, for every scalar $\lambda > 0$, $\lambda x = (\lambda x_1, \lambda x_2)$ and

$$g_2(\lambda x) = \lambda g_2(x),\ \forall x \in \mathbb{R}^2,$$

i.e., $g_2$ is homogeneous of degree 1. Hence, $g_2$ is *concave* on $\mathbb{R}^2$. Analytically, this means that $\forall x, y \in \mathbb{R}^2$, and $0 \le \lambda \le 1$, then

$$\lambda g_2(x) + (1-\lambda) g_2(y) \le g_2(\lambda x + (1-\lambda) y).$$

To get an idea of what it means, geometrically, try and picture the surface in 3-space whose equation is

$$z = \sqrt{xy},\ x, y \ge 0.$$

**Exercise 4.** *Apropos Problem 2, draw a similar conclusion about the function*

$$g_3(x) = \sqrt[3]{x_1 x_2 x_3},\ x = (x_1, x_2, x_3) \in \mathbb{R}^3,\ \ x_i \ge 0, i = 1, 2, 3.$$

In fact, if $x = (x_1, x_2, \ldots, x_n)$ is a vector in $n$-space and its coordinates are positive, and

$$g_n(x) = \sqrt[n]{x_1 x_2 \cdots x_n},$$

then $g_n$ is *super-additive and is homogeneous of degree 1* on

$$\{x \in \mathbb{R}^n : x_i \ge 0,\ i = 1, 2, \ldots, n\}.$$

Hence, $g_n$ is concave on this region.

## 5 Discussion of Theorem 5

It's familiar that $\sqrt{a^2+b^2}$ is the euclidean distance from the origin to the point $(a,b)$ in $\mathbb{R}^2$. Thus Theorem 5 is simply an analytical statement equivalent to the triangle inequality! To put this another way, let

$$\ell_2(x) = \sqrt{x_1^2 + x_2^2},\ x = (x_1, x_2) \in \mathbb{R}^2,$$

then

$$\ell_2(x+y) \le \ell_2(x) + \ell_2(y),\ \forall x, y \in \mathbb{R}^2.$$

In other words, $\ell_2$ is *sub-additive.* But it's also homogeneous of degree one. Hence, it is *convex* on $\mathbb{R}^2$, i.e., $\forall x, y \in \mathbb{R}^2$, and $0 \le \lambda \le 1$, then

$$\ell_2(\lambda x + (1-\lambda)y) \le \lambda \ell_2(x) + (1-\lambda)\ell_2(y).$$

To get an idea of what this means, geometrically, try and picture the region of points $(x,y,z)$ in 3-space constrained by the conditions

$$0 \le z \le \sqrt{x^2+y^2},\ x, y \in \mathbb{R}.$$

[Hint: at your next meal examine the sugar bowl, say.]
A solution of Problem 1 follows once we can show that

$$\ell_3(x) = \sqrt{x_1^2 + x_2^2 + x_3^2},\ x = (x_1, x_2, .x_3) \in \mathbb{R}^3,$$

has the same property.

**Exercise 5.** *Establish that $\ell_3$ is sub-additive and convex on $\mathbb{R}^3$, and solve Problem 1.*

## 6 Solution of Problem 3

By now it will have occurred to you that what is wanted to solve Problem 3 is an analogue of Corollary 1 for the *harmonic mean* of three positive numbers. (This has properties similar to the geometric mean, but doesn't get as much attention, and doesn't feature in too many IMO problems.) Namely, we want to show that the function $h_3$ defined by

$$h_3(x) = \frac{3}{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}},\ x = (x_1, x_2, x_3) \in \mathbb{R}^3,\ x_1, x_2, x_3 > 0$$

is super-additive, and so concave, on $\mathbb{R}^3$.

**Theorem 7.** *If $a, b, c > 0$, then*

$$\frac{1}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} = \min\{ax + by + cz : 0 < x, y, z, \sqrt{x} + \sqrt{y} + \sqrt{z} = 1\}.$$

Proof. This is a consequence of Theorem 4. For, if $x, y, z > 0$ and $\sqrt{x}+\sqrt{y}+\sqrt{z} = 1$, then

$$1 = (\sqrt{ax}\frac{1}{\sqrt{a}} + \sqrt{by}\frac{1}{\sqrt{b}} + \sqrt{cz}\frac{1}{\sqrt{c}})^2 \leq (ax + by + cz)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}).$$

Hence,

$$\frac{1}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \leq ax + by + cz,$$

whenever $0 < x, y, z$, $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$, and there is equality here when

$$x = \frac{1}{a^2(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})^2}, \quad y = \frac{1}{b^2(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})^2}, \quad z = \frac{1}{c^2(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})^2}.$$

**Corollary 3.** *If $a, b, c, d, e, f > 0$, then*

$$h_3(a, b, c) + h_3(d, e, f) \leq h_3(a + d, b + e, c + f).$$

*There is equality iff*

$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f}.$$

Proof. For, if $x, y, z > 0$ and $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$, then

$$h_3(a, b, c)+h_3(d, e, f) \leq 3(ax+by+cz)+3(dx+ey+fz) = 3[(a+d)x+(b+e)y+(c+f)z],$$

and so $h_3(a, b, c) + h_3(d, e, f)$ doesn't exceed

$$3 \min\{(a+d)x+(b+e)y+(c+f)z : 0 < x, y, z, \sqrt{x}+\sqrt{y}+\sqrt{z} = 1\} = h_3(a+d, b+e, c+f).$$

Moreover, (?) there is equality here iff

$$\sqrt{x} = \frac{1}{a(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})} = \frac{1}{d(\frac{1}{d} + \frac{1}{e} + \frac{1}{f})} = \frac{1}{(a + d)(\frac{1}{a+d} + \frac{1}{b+e} + \frac{1}{c+f})},$$

$$\sqrt{y} = \frac{1}{b(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})} = \frac{1}{e(\frac{1}{d} + \frac{1}{e} + \frac{1}{f})} = \frac{1}{(b + e)(\frac{1}{a+d} + \frac{1}{b+e} + \frac{1}{c+f})},$$

and

$$\sqrt{z} = \frac{1}{c(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})} = \frac{1}{f(\frac{1}{d} + \frac{1}{e} + \frac{1}{f})} = \frac{1}{(c + f)(\frac{1}{a+d} + \frac{1}{b+e} + \frac{1}{c+f})}.$$

In other words, (?) iff

$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f}.$$

**Exercise 6.** *Now solve Problem 3.*

# 7 Exercises

1. Suppose $a, b > 0$. Prove that

$$\frac{a + g_2(a,b)}{2} \leq g_2(a, \frac{a+b}{2}),$$

with equality iff $a = b$.

2. State and prove an analogue of Theorem 7 and its corollary for $h_2$, where

$$h_2(x) = \frac{2}{\frac{1}{x_1} + \frac{1}{x_2}}, \; x = (x_1, x_2) \in \mathbb{R}^2, \; x_i, x_2 > 0.$$

3. Suppose $a, b > 0$. Prove that

$$\frac{a + h_2(a,b)}{2} \leq h_2(a, \frac{a+b}{2}),$$

with equality iff $a = b$.

4. Suppose $a, b, c > 0$. Prove that

$$g_2(a,b) + g_2(b,c) + g_2(c,a) \leq a + b + c,$$

with equality iff $a = b = c$.

5. Suppose $a, b, c > 0$. Prove that

$$h_2(a,b) + h_2(b,c) + h_2(c,a) \leq a + b + c,$$

with equality iff $a = b = c$.

6. Suppose $a, b, c > 0$. Prove that

$$3\sqrt[3]{abc} = \min\{ax + by + cz : 0 < x, y, z, xyz = 1\}.$$

Deduce Corollary 1.

7. More generally, if $x_i \geq 0$, $i = 1, 2, \ldots, n$, $x = (x_1, x_2, \ldots, x_n)$, and

$$g_n(x) = \sqrt[n]{x_1 x_2 \cdots x_n}.$$

Prove that

$$ng_n(x) = \min\{\sum_{i=1}^{n} x_i y_i : y_1 y_2 \cdots y_n = 1, \; y_i > 0, i = 1, 2 \ldots, n\}.$$

Now redo Exercise 2 in Section 2.

8. Suppose $a, b, c > 0$. Prove that

$$a + g_2(a, b) + g_3(a, b, c) \leq 3g_3(a, \frac{a+b}{2}, \frac{a+b+c}{3}),$$

with equality iff $a = b = c$.

9. Suppose $a, b, c > 0$. Prove that

$$a + h_2(a, b) + h_3(a, b, c) \leq 3h_3(a, \frac{a+b}{2}, \frac{a+b+c}{3}),$$

with equality iff $a = b = c$.

10. Suppose $a, b, c, d > 0$. Prove that

$$\frac{ab}{a+b+1} + \frac{cd}{c+d+1} < \frac{(a+c)(b+d)}{a+b+c+d+1}.$$

11. Suppose $a, b, c > 0$. Prove that

$$g_2(a, b) + g_2(b, c) + g_2(c, a) \leq 3g_3(\frac{a+b}{2}, \frac{b+c}{2}, \frac{c+a}{2}) \leq a + b + c,$$

with equality iff $a = b = c$.

12. Suppose $a, b, c > 0$. Prove that

$$h_2(a, b) + h_2(b, c) + h_2(c, a) \leq 3h_3(\frac{a+b}{2}, \frac{b+c}{2}, \frac{c+a}{2}) \leq a + b + c,$$

with equality iff $a = b = c$.