Solutions to some Inequality problems
Lecture II, UL 2007

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1 Solution of Exercise 6

Theorem 1. *Suppose $a, b, c > 0$. Then*

$$3\sqrt[3]{abc} = \min\{ax + by + cz : 0 < x, y, z, xyz = 1\}.$$

Proof. By the AM-GM inequality, if $0 < x, y, z$ and $xyz = 1$, then

$$3\sqrt[3]{abc} = 3\sqrt[3]{(ax)(by)(cz)} \leq ax + by + cz,$$

with equality iff

$$ax = by = cz = \sqrt[3]{abc}.$$

More generally, a similar proof shows that

Theorem 2. *Suppose $a_i > 0, i = 1, 2, \ldots, n$. Then*

$$n\sqrt[n]{a_1 a_2 \cdots a_n} = \min\{\sum_{i=1}^{n} a_i x_i : 0 < x_i, x_1 x_2 \cdots x_n = 1\}.$$

2 Solution of Exercise 8

Problem. *Suppose $a, b, c > 0$. Prove that*

$$a + g_2(a, b) + g_3(a, b, c) \leq 3g_3(a, \frac{a+b}{2}, \frac{a+b+c}{3}),$$

with equality iff $a = b = c$.

Solution. We'll utilise Exercise 6., which we've just established.
Let $x, y, z > 0, xyz = 1$. Let $p = \sqrt[3]{x^2y},\ q = \sqrt[3]{yz^2}$. Then $pq = 1$ and so

$$2\sqrt{ab} \le pa + qb,\ \ 3\sqrt[3]{abc} \le xa + yb + zc,$$

whence

$$\begin{aligned} a + \sqrt{ab} + \sqrt[3]{abc} &\le a + (pa + qb)/2 + (xa + yb + zc)/3 \\ &= (1 + p/2 + x/3)a + (q/2 + y/3)b + (z/3)c \\ &\le (x + y/2 + z/3)a + (y/2 + z/3)b + cz/3 \\ &= xa + y(a + b)/2 + z(a + b + c)/3 \end{aligned}$$

provided

$$1 + p/2 + x/3 \le x + y/2 + z/3,\ \ q/2 + y/3 \le y/2 + z/3.$$

Now

$$\begin{aligned} 1 + p/2 + x/3 &\le (x + y + z)/3 + \sqrt[3]{x^2y}/2 + x/3 \\ &\le \frac{2x + y + z + x + y/2}{3} \\ &= x + \frac{3y/2 + z}{3} \\ &= x + y/2 + z/3, \end{aligned}$$

with equality iff $x = y = z = 1$. Also,

$$\begin{aligned} q/2 + y/3 &= \sqrt[3]{yz^2}/2 + y/3 \\ &\le \frac{y/2 + z + y}{3} \\ &= y/2 + z/3, \end{aligned}$$

with equality iff $y = z$. It follows that

$$a+\sqrt{ab}+\sqrt[3]{abc} \le \min\{xa+y(\frac{a+b}{2})+z(\frac{a+b+c}{3}) : x, y, z > 0, xyz = 1\} = 3g_3(a, \frac{a+b}{2}, \frac{a+b+c}{3}).$$

A slicker approach is to use the super-additivity property of the geometric means.
Indeed, since

$$g_2(a, b) = g_3(a, g_2(a, b), b),$$

we have

$$\begin{aligned} a + g_2(a, b) + g_3(a, b, c) &= g_3(a, a, a) + g_3(a, g_2(a, b), b) + g_3(a, b, c) \\ &\le g_3(a + a + a, a + g_2(a, b) + b, a + b + c) \\ &\le g_3(3a, \frac{3(a + b)}{2}, a + b + c0 \\ &= \sqrt[3]{(3a)(\frac{3(a + b)}{2})\frac{3(a + b + c)}{3}} \\ &= 3g_3(a, \frac{a + b}{2}, \frac{a + b + c}{3}). \end{aligned}$$

In arriving at this result we've used the fact that

$$g_2(a,b) \le \frac{a+b}{2}$$

and the fact that g_3 is an increasing function of each of its arguments.
Remark You should examine the case of equality.
As a consequence, we have that

$$a + g_2(a,b) + g_3(a,b,c) \le \frac{3}{\sqrt[3]{3!}}(a+b+c).$$

More generally, show that if $a_i > 0,\ i = 1, 2, \ldots, n$, and

$$b_k = \sqrt[k]{a_1 a_2 \cdots a_k}, k = 1, 2, \ldots, n,$$

then

$$\sum_{k=1}^{n} b_k \le \frac{n}{\sqrt[n]{n!}} \sum_{k=1}^{n} a_k.$$

3 Solution of Exercise 9

Problem. *Suppose $a, b, c > 0$. Prove that*

$$a + h_2(a,b) + h_3(a,b,c) \le 3h_3(a, \frac{a+b}{2}, \frac{a+b+c}{3}),$$

with equality iff $a = b = c$.

A similar strategy to that used in the previous problem can be employed to deal with this by using Theorem 7.
Proof. Suppose $x, y, z > 0, x + y + z = 1$. Choose $p = x + y/2, q = y/2 + z$. Consider $a + 2(p^2 a + q^2 b) + 3(x^2 a + y^2 b + z^2 c)$. I claim that this is dominated by $9[x^2 a + y^2(a+b)/2 + z^2(a+b+c)/3]$. This is the case provided that

$$1 + 2p^2 + 3x^2 \le 9(x^2 + y^2/2 + z^2/3),\ 2q^2 + 3y^2 \le 9(y^2/2 + z^2/3).$$

Now $1 \le 3(x^2 + y^2 + z^2)$. Hence

$$\begin{aligned} 1 + 2p^2 + 3x^2 &\le 6x^2 + 3y^2 + 3z^2 + 2(x + y/2)^2 \\ &= 8x^2 + 7y^2/2 + 2xy + 3z^2 \\ &\le 9x^2 + 9y^2/2 + 3z^2 \\ &= 9(x^2 + y^2/2 + z^2/3) \end{aligned}$$

with equality iff $x = y = z = 1/3$. Also

$$\begin{aligned} 2q^2 + 3y^2 &= 2(y/2 + z)^2 + 3y^2 \\ &= 7y^2/2 + 2yz + 2z^2 \\ &\le 9y^2/2 + 3z^2 \end{aligned}$$

with equality iff $y = z$. Since $p + q = 1$, we have that

$$\begin{aligned} a + \frac{2}{\frac{1}{a}+\frac{1}{b}} + \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} &\leq a + 2(p^2a + q^2b) + 3(x^2a + y^2b + z^2c) \\ &= (1 + 2p^2 + 3x^2)a + (2q^2 + 3y^2)b + 3z^2c \\ &\leq 9(x^2 + y^2/2 + z^2/3)a + (y^2/2 + z^2/3)b + z^3c/3) \\ &= 9[x^2a + y^2(a + b)/2 + z^2(a + b + c)/3], \end{aligned}$$

whence

$$a + \frac{2}{\frac{1}{a}+\frac{1}{b}} + \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$$

is not bigger than any element in the set

$$\{9[x^2a + y^2(a + b)/2 + z^2(a + b + c)/3] : x + y + z = 1\},$$

whose minimum, by Theorem 7, is

$$\frac{9}{\frac{1}{a}+\frac{2}{a+b}+\frac{3}{a+b+c}}.$$

This completes the solution.

Exercise 1. *Establish Exercise 9 by using the additivity property of the harmonic means.*

Since $a < a + b < a + b + c$, we have that

$$\frac{9}{\frac{1}{a}+\frac{2}{a+b}+\frac{3}{a+b+c}} < \frac{3}{2}(a + b + c).$$

Hence, we can infer from this problem that

$$a + h_2(a, b) + h_3(a, b, c) < \frac{3}{2}(a + b + c).$$

4 Solution of Problem 10

Problem. *Suppose $a, b, c, d > 0$. Prove that*

$$\frac{ab}{a + b + 1} + \frac{cd}{c + d + 1} < \frac{(a + c)(b + d)}{a + b + c + d + 1}.$$

This is one of the Monthly problems posted on the University of Purdue site. It was drawn to my attention by Prithwijit De.
Solution. With

$$f(x, y) = \frac{xy}{x + y + 1},\ x, y \geq 0,$$

what we want to show is that

$$f(a,b)+f(c,d)<f(a+c,b+d),$$

i.e., that f is super-additive. By Corollary 3, Section 6,

$$\begin{aligned}\frac{ab}{a+b+1}+\frac{cd}{c+d+1} &= \frac{1}{\frac{1}{a}+\frac{1}{b}+\frac{1}{ab}}+\frac{1}{\frac{1}{c}+\frac{1}{d}+\frac{1}{cd}}\\ &\le \frac{1}{\frac{1}{a+c}+\frac{1}{b+d}+\frac{1}{ab+cd}}\\ &< \frac{1}{\frac{1}{a+c}+\frac{1}{b+d}+\frac{1}{(a+c)(b+d}}\\ &= \frac{(a+c)(b+d)}{a+b+c+d+1},\end{aligned}$$

since

$$ab+cd<(a+c)(b+d),\ \frac{1}{ab+cd}>\frac{1}{(a+c)(b+d)}$$

and, for $s\ge 0$, $t\to s+\frac{1}{t}$ is a strictly increasing function of $t>0$.

5 Solution of Exercise 11

Problem (Carlson, 1971). *Suppose* $x,y,z\ge 0$. *Then*

$$\frac{\sqrt{xy}+\sqrt{yz}+\sqrt{zx}}{3}\le\sqrt[3]{\frac{x+y}{2}\frac{y+z}{2}\frac{z+x}{2}},$$

with equality iff $x=y=z$.

Solution. Replace x,y,z by a^2,b^2,c^2, and put $t=a^2+b^2+c^2$. Then we have to prove that

$$\begin{aligned}\frac{8}{27}(ab+bc+ca)^3 &\le (t-a^2)(t-b^2)(t-c^2)\\ &= t^3-t^2(a^2+b^2+c^2)+t(a^2b^2+b^2c^2+c^2a^2)-a^2b^2c^2\\ &= (a^2+b^2+c^2)(a^2b^2+b^2c^2+c^2a^2)-a^2b^2c^2,\end{aligned}$$

i.e., the stated inequality is equivalent to the following one:

$$\frac{8}{27}(ab+bc+ca)^3+a^2b^2c^2\le(a^2+b^2+c^2)(a^2b^2+b^2c^2+c^2a^2),$$

OR

$$\frac{8}{27}(\sqrt{xy}+\sqrt{yz}+\sqrt{zx})^3+xyz\le(x+y+z)(xy+yz+zx),$$

with equality iff $x=y=z$.

Now

$$\begin{aligned}
a^2b^2c^2 &= \sqrt[3]{a^2b^2c^2}\ \sqrt[3]{(a^2b^2)(b^2c^2)(c^2a^2)} \\
&\le \frac{a^2+b^2+c^2}{3}\ \frac{a^2b^2+b^2c^2+c^2a^2}{3} \\
&= \frac{1}{9}(a^2+b^2+c^2)(a^2b^2+b^2c^2+c^2a^2),
\end{aligned}$$

and there is equality iff $a^2 = b^2 = c^2$. Next

$$ab+bc+ca \le a^2+b^2+c^2,\ ab+bc+ca \le \sqrt{a^2b^2+b^2c^2+c^2a^2}\sqrt{3},$$

with equality in both iff $a = b = c$. Hence

$$(ab+bc+ca)^3 \le 3(a^2+b^2+c^2)(a^2b^2+b^2c^2+c^2a^2),$$

with equality iff $a = b = c$.
Combining these results we deduce that

$$\begin{aligned}
\frac{8}{27}(ab+bc+ca)^3 + a^2b^2c^2 &\le \frac{8}{9}(a^2+b^2+c^2)(a^2b^2+b^2c^2+c^2a^2) + \frac{1}{9}(a^2+b^2+c^2)(a^2b^2+b^2c^2+c^2a^2) \\
&= (a^2+b^2+c^2)(a^2b^2+b^2c^2+c^2a^2),
\end{aligned}$$

with equality iff $a = b = c$. The result follows.
By analogy, one is tempted to suggest that the following statement is true.

Problem. *Suppose $x, y, z \ge 0$. Then*

$$\frac{h_2(x,y)+h_2(y,z)+h_2(z,x)}{3} \le h_3(\frac{x+y}{2}, \frac{y+z}{2}, \frac{z+x}{2}),$$

with equality iff $x = y = z$.

6 IMO 1988, Problem 4

Problem. *Show that the set of real numbers x which satisfy the inequality*

$$\sum_{k=1}^{70} \frac{k}{x-k} \ge \frac{5}{4},$$

is a union of disjoint intervals the sum of whose lengths is 1988.

Solution. Draw the graph of

$$f(x) = \sum_{k=1}^{70} \frac{k}{x-k},$$

and denote the x coordinates of the points where it crosses the horizontal line $y = 5/4$ by $a_i, i = 1, 2, \ldots .70$. The set $\{x : f(x) \geq 5/4\}$ is the union of the intervals,

$$(i, a_i],\ \ i = 1, 2, \ldots, 70,$$

the sum of whose lengths is

$$\sum_{i=1}^{70}(a_i - i).$$

To determine this sum, note that the a_i are the roots of the polynomial

$$\prod_{k=1}^{70}(x-k)(f(x)-\frac{5}{4}) = 0.$$

Pick out the coefficient of x^{69}; this determines the sum of the $a_i - i$. From this one can answer the problem.
The problem posed is a special case of the following.

Problem (Boole, Loomis). *Suppose $a_i, i = 1, 2, \ldots, n$ are real numbers and $m_i, i = 1, 2, \ldots, n$ are positive numbers. Let $\lambda > 0$. Prove that the set*

$$\{x : \sum_{k=1}^{n} \frac{m_i}{x - a_i} \geq \lambda\}$$

is a union of disjoint intervals the sum of whose lengths is

$$\frac{1}{\lambda}\sum_{i=1}^{n} m_i.$$