6. Theorem of Ceva, Menelaus and Van Aubel.

Theorem 1 *(Menelaus). If A_1, B_1, C_1 are points on the sides BC, CA and AB of a triangle ABC, then the points are collinear if and only if*

$$\frac{|A_1B|}{|A_1C|}.\frac{|B_1C|}{|B_1A|}.\frac{|C_1A|}{|C_1B|} = 1.$$

Proof Assume points are collinear.

First drop perpendiculars AA', BB' and CC' from the vertices A, B, C to the line $A_1B_1C_1$. Then since AA', BB' and CC' are perpendicular to A_1B_1, they are parallel (Figure 1). Thus we get the following equalities of ratios

$$\frac{|A_1B|}{|A_1C|} = \frac{|BB'|}{|CC'|}, \quad \frac{|B_1C|}{|B_1A|} = \frac{|CC'|}{|AA'|}$$

$$\text{and} \quad \frac{|C_1A|}{|C_1B|} = \frac{|AA'|}{|BB'|}.$$

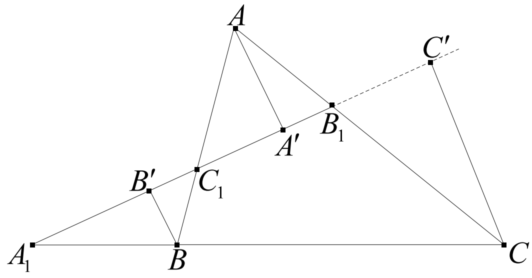

Figure 1:

Multiplying these we get the required result.

Conversely, suppose $\frac{|A_1B|}{|A_1C|}.\frac{|B_1C|}{|B_1A|}.\frac{|C_1A|}{|C_1B|} = 1.$

Now suppose lines BC and B_1C_1 meet at the point A''. Then

$$\frac{|A''B|}{|A''C|}.\frac{|B_1C|}{|B_1A|}.\frac{|C_1A|}{|C_1B|} = 1.$$

$$\text{Thus} \quad \frac{|A_1B|}{|A_1C|} = \frac{|A''B|}{|A''C|},$$

and so we conclude that the point A'' on the line BC coincides with the point A_1. Thus the points A_1, B_1 and C_1 are collinear.

Definition 1 *A line segment joining a vertex of a triangle to any given point on the opposite side is called a Cevian.*

Theorem 2 *(Ceva) Three Cevians AA_1, BB_1 and CC_1 of a triangle ABC (Figure 2) are concurrent if and only if*

$$\frac{|BA_1|}{|A_1C|} \cdot \frac{|CB_1|}{|B_1A|} \cdot \frac{|AC_1|}{|C_1B|} = 1.$$

Proof First assume that the Cevians are concurrent at the point M.

Consider the triangle AA_1C and apply Menelaus' theorem. Since the points B_1, M and B are collinear,

$$\frac{|B_1C|}{|B_1A|} \cdot \frac{|MA|}{|MA_1|} \cdot \frac{|BA_1|}{|BC|} = 1 \qquad \ldots (a)$$

Now consider the triangle AA_1B. The points C_1, M, C are collinear so

$$\frac{|C_1A|}{|C_1B|} \cdot \frac{|CB|}{|CA_1|} \cdot \frac{|MA_1|}{|MA|} = 1 \qquad \ldots (b)$$

Multiply both sides of equations (a) and (b) to get required result.

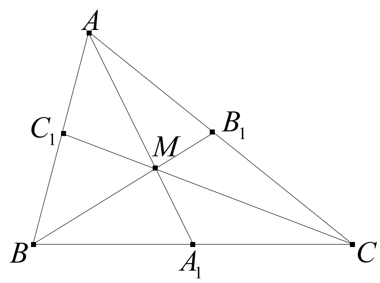

Figure 2:

Conversely, suppose the two Cevians AA_1 and BB_1 meet at P and that the Cevian from the vertex C through P meets side AB at C'. Then we have

$$\frac{|BA_1|}{|A_1C|} \cdot \frac{|CB_1|}{|B_1A|} \cdot \frac{|AC'|}{|C'B|} = 1.$$

By hypothesis,

$$\frac{|BA_1|}{|A_1C|} \cdot \frac{|CB_1|}{|B_1A|} \cdot \frac{|AC_1|}{|C_1B|} = 1.$$

$$\text{Thus} \qquad \frac{|AC_1|}{|C_1B|} = \frac{|AC'|}{|C'B|},$$

and so the two points C_1 and C' on the line segment AB must coincide. The required result follows.

Theorem 3 *(van Aubel) If A_1, B_1, C_1 are interior points of the sides BC, CA and AB of a triangle ABC and the corresponding Cevians AA_1, BB_1 and CC_1 are concurrent at a point M (Figure 3), then*

$$\frac{|MA|}{|MA_1|} = \frac{|C_1A|}{|C_1B|} + \frac{|B_1A|}{|B_1C|}.$$

Proof Again, as in the proof of Ceva's theorem, we apply Menelaus' theorem to the triangles AA_1C and AA_1B.

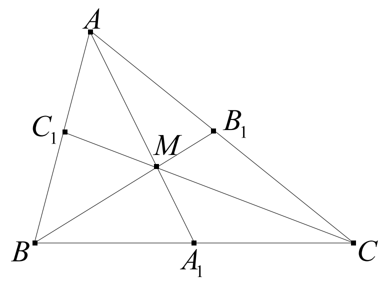

Figure 3:

In the case of AA_1C, we have

$$\frac{|B_1C|}{|B_1A|}.\frac{|MA|}{|MA_1|}.\frac{|BA_1|}{|BC|} = 1,$$

and so

$$\frac{|B_1A|}{|B_1C|} = \frac{|MA|}{|MA_1|}.\frac{|BA_1|}{|BC|} \qquad \ldots (c)$$

For the triangle AA_1B, we have

$$\frac{|C_1A|}{|C_1B|}.\frac{|CB|}{|CA_1}.\frac{|MA_1|}{|MA|} = 1,$$

and so

$$\frac{|C_1A|}{|C_1B|} = \frac{|MA|}{|MA_1|}.\frac{|CA_1|}{|BC|} \qquad \ldots (d)$$

Adding (c) and (d) we get

$$\frac{|B_1A|}{|B_1C|} + \frac{|C_1A|}{|C_1B|} = \frac{|MA|}{|MA_1||BC|}\{|BA_1| + |A_1C|\} \qquad = \frac{|MA|}{|MA_1|},$$

as required.

Examples

1. Medians AA_1, BB_1 and CC_1 intersect at the centroid G and then

$$\frac{|GA|}{|GA_1|} = 2,$$

since

$$1 = \frac{|A_1B|}{|A_1C|} = \frac{|B_1C|}{|B_1A|} = \frac{|C_1A|}{|C_1B|}.$$

2. The angle bisectors in a triangle are concurrent at the incentre I of the triangle. Furthermore, if A_3, B_3 and C_3 are the points on the sides BC, CA and AB where the bisectors intersect these sides (Figure 4), then

$$\frac{|A_3B|}{|A_3C|} = \frac{c}{b}, \frac{|B_3C|}{|B_3A|} = \frac{a}{c} \text{ and} \frac{|C_3A|}{|C_3B|} = \frac{b}{a}.$$

Then

$$\begin{aligned} \frac{|IA|}{|IA_3|} &= \frac{|C_3A|}{|C_3B|} + \frac{|B_3A|}{|B_3C|} \\ &= \frac{b}{a} + \frac{c}{a} = \frac{b+c}{a}. \end{aligned}$$

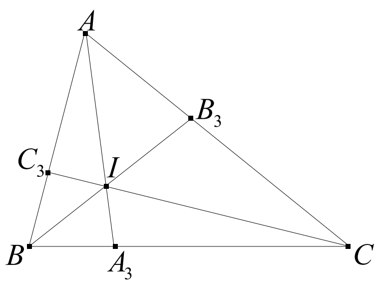

Figure 4:

3. Let AA_2, BB_2 and CC_2 be the altitudes of a triangle ABC. They are concurrent at H, the orthocentre of ABC (Figure 5.)

We have

$$\begin{aligned} \frac{|A_2B|}{|A_2C|} &= \frac{|AA_2|\cot(\widehat{B})}{|AA_2|\cot(\widehat{C})} \\ &= \frac{\tan(\widehat{C})}{\tan(\widehat{B})} \end{aligned}$$

and similarly

$$\begin{aligned} \frac{|B_2C|}{|B_2A|} &= \frac{\tan(\widehat{A})}{\tan(\widehat{C})}, \\ \frac{|C_2A|}{|C_2B|} &= \frac{\tan(\widehat{B})}{\tan(\widehat{C})}. \end{aligned}$$

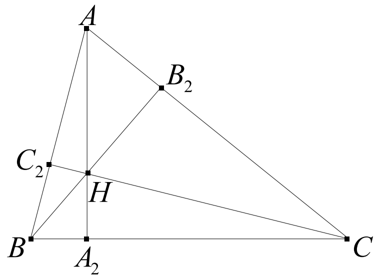

Figure 5:

Multiplying the 3 ratios, we get concurrency of the altitudes. Furthermore,

$$\frac{|HA|}{|HA_2|} = \frac{|C_2A|}{|C_2B|} + \frac{|B_2A|}{|B_2C|} = \frac{\tan(\widehat{B})}{\tan(\widehat{A})} + \frac{\tan(\widehat{C})}{\tan(\widehat{A})}$$
$$= \frac{\tan(\widehat{B}) + \tan(\widehat{C})}{\tan(\widehat{A})}$$
$$= \frac{\sin(\widehat{B} + \widehat{C}).\cos(\widehat{A})}{\cos(\widehat{B})\cos(\widehat{C})\sin(\widehat{A})}$$
$$= \frac{\sin(180^\circ - \widehat{A})\cos(\widehat{A})}{\cos(\widehat{B})\cos(\widehat{C})\sin(\widehat{A})} = \frac{\cos(\widehat{A})}{\cos(\widehat{B})\cos(\widehat{C})}.$$

Lemma 1 *Let ABC be a triangle and A_1 a point on the side BC so that*

$$\frac{|A_1B|}{|A_1C|} = \frac{\gamma}{\beta}$$

Let X and Y be points on the sides AB and AC respectively and let M be the point of intersection of the line segments XY and AA_1 (Figure 6). Then

$$\beta(\frac{|XB|}{|XA|}) + \gamma(\frac{|YC|}{|YA|}) = (\beta + \gamma)(\frac{|A_1M|}{|MA|}).$$

Proof First suppose that XY is parallel to the side BC. Then

$$\frac{|XB|}{|XA|} = \frac{|YC|}{|YA|} = \frac{|MA_1|}{|MA|},$$

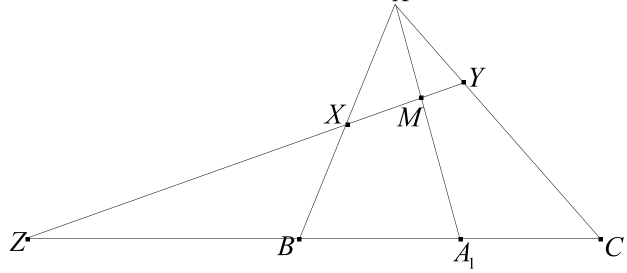

Figure 7:

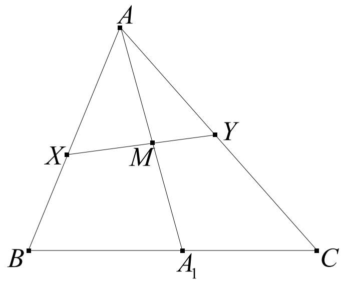

Figure 6:

and so result is true for any β and γ.

Now suppose the lines XY and BC intersect at a point Z.

Consider the triangle AA_1B (Figure 7). Since M, X and Z are collinear,

$$\frac{|YC|}{|YA|}.\frac{|MA|}{|MA_1|}.\frac{|ZA_1|}{|ZC|} = 1.$$

$$\begin{aligned}
\text{Then} \quad & \beta(\frac{|XB|}{|XA|}) + \gamma(\frac{|YC|}{|YA|}) \\
= \quad & \beta(\frac{|MA_1||ZB|}{|MA||ZA_1|}) + \gamma(\frac{|MA_1||ZC|}{|MA||ZA_1|}) \\
= \quad & \frac{|MA_1|}{|MA||ZA_1|}\{\beta|ZB| + \gamma|ZC|\} \\
= \quad & \frac{|MA_1|}{|MA||ZA_1|}\{\beta|ZA_1| - \beta|BA_1| + \gamma|ZA_1| + \gamma|A_1C|\} \\
= \quad & (\beta+\gamma)\frac{|MA_1|}{|MA||ZA_1|}.|ZA_1|, \\
& \text{since } \frac{|BA_1|}{|A_1C|} = \frac{\gamma}{\beta}, \\
= \quad & (\beta+\gamma)\frac{|MA_1|}{|MA|}, \qquad \text{as required.}
\end{aligned}$$

Theorem 4 *Let ABC be a triangle with three cevians AA_1, BB_1 and CC_1 intersecting at a point M (Figure 8).*

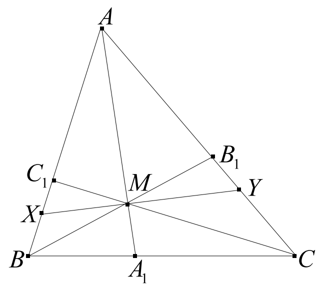

Figure 8:

Furthermore suppose

$$\frac{|A_1B|}{|A_1C|} = \frac{\gamma}{\beta}, \ \frac{|B_1C|}{|B_1A|} = \frac{\alpha}{\gamma} \ \ \textit{and} \ \ \frac{|C_1A|}{|C_1B|} = \frac{\beta}{\alpha}.$$

If X and Y are points on the sides AB and AC then the point M belongs to the line segment XY if and only if

$$\beta(\frac{|XB|}{|XA|}) + \gamma(\frac{|YC|}{|YA|}) = \alpha.$$

Proof By van Aubel's theorem:

$$\begin{aligned} \frac{|AM|}{|A_1M|} &= \frac{|C_1A|}{|C_1B|} + \frac{|B_1A|}{|B_1C|} \\ &= \frac{\beta}{\alpha} + \frac{\gamma}{\alpha} = \frac{\beta+\gamma}{\alpha}. \end{aligned}$$

Now suppose M belongs to the line segment XY. Then by the previous lemma

$$\begin{aligned} \beta(\frac{|XB|}{|XA|}) + \gamma(\frac{|YC|}{|YA|}) &= (\beta+\gamma)\frac{|A_1M|}{|MA|} \\ &= (\beta+\gamma)(\frac{\alpha}{\beta+\gamma}) = \alpha, \quad \text{as required.} \end{aligned}$$

For converse, suppose XY and AA_1 intersect in point M'. We will show that M' coincides M.

By the lemma,

$$\beta(\frac{|XB|}{|XA|}) + \gamma(\frac{|YC|}{|YA|}) = (\beta+\gamma)(\frac{|A_1M'|}{|M'A|}).$$

By hypothesis, we have

$$\beta(\frac{|XB|}{|XA|}) + \gamma(\frac{|YC|}{|YA|}) = \alpha.$$

Thus

$$\frac{|A_1M|}{|AM|} = \frac{\alpha}{\beta+\gamma},$$

and so M and M' coincide. Thus M must lie on the line segment XY.

Corollary 1 *If G is the centroid of the triangle ABC and so $\alpha = \beta = \gamma = 1$, then G belongs to the line segment XY if and only if*

$$\frac{|XB|}{|XA|} + \frac{|YC|}{|YA|} = 1.$$

Corollary 2 *If I is the incentre of the triangle ABC then the values of α, β and γ are given in terms of the sidelengths of the triangle as*

$$\alpha = a, \quad \beta = b \quad \text{and} \quad \gamma = c.$$

Thus I belongs to XY if and only if

$$b(\frac{|XB|}{|XA|}) + c(\frac{|YC|}{|YA|}) = a.$$

Corollary 3 *If H is the orthocentre of the triangle ABC then the ratios on the sides are given by*

$$\alpha = \tan(\widehat{A}), \quad \beta = \tan(\widehat{B}) \quad \text{and} \quad \gamma = \tan(\widehat{C}.)$$

Then we get that H belongs to the line segment XY if and only if

$$(\tan(\widehat{B}))(\frac{|XB|}{|XA|}) + (\tan(\widehat{C}))(\frac{|YC|}{|YA|}) = \tan(\widehat{A}).$$

We also get the following result which was a question on the 2006 Irish Invervarsity Mathematics Competition.

Theorem 5 *Let ABC is a triangle and let X and Y be points on the sides AB and AC respectively such that the line segment XY bisects the area of ABC and the points X and Y bisects the perimeter (Figure 9). Then the incentre I belongs to the line segment XY.*

Proof Let $x = |AX|$ and $y = |AY|$.

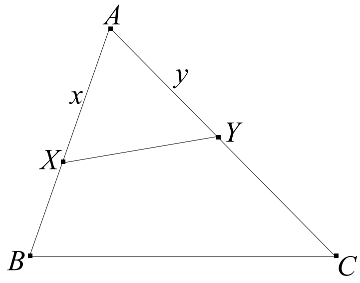

Figure 9:

Then

$$x + y = \frac{a+b+c}{2} \qquad \ldots (a)$$

where a, b and c are lengths of sides.

Furthermore,

$$\frac{1}{2} = \frac{area(AXY)}{area(ABC)} = \frac{xy\sin(\widehat{A})}{bc\sin(\widehat{A})},$$

so

$$xy = \frac{bc}{2} \qquad \ldots (b).$$

$$\text{Consider} \qquad b(\frac{|XB|}{|XA|}) + c(\frac{|YC|}{|YA|})$$
$$= b(\frac{c-x}{x}) + c(\frac{b-y}{y})$$
$$= b(\frac{1}{x} + \frac{1}{y}) - b - c$$
$$= bc(\frac{a+b+c}{2}.\frac{2}{bc}) - b - c$$
$$= a.$$

Thus by Corollary 2, incentre I belongs to the line XY.

Theorem 6 *Let ABC be an equilateral triangle and X, Y and Z points on the sides BC, CA and AB respectively (Figure 10). Then the minimum value of*

$$|ZX|^2 + |XY|^2 + |YZ|^2$$

is attained when X, Y, Z are the midpoints of the sides.

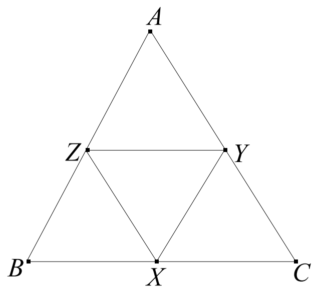

Figure 10:

Proof Consider $\frac{1}{3}\{|ZX|^2+|XY|^2+|YZ|^2\}$

$$\begin{aligned} \text{We have} \quad & \frac{1}{3}\{|ZX|^2+|XY|^2+|YZ|^2\} \\ & \geq (\frac{|ZX|+|XY|+|YZ|}{2})^2, \\ & \text{by Cauchy} - \text{Schwarz inequality,} \\ & \geq (\frac{|A_1B_1|+|B_1C_1|+|C_1A_1|}{3})^2, \end{aligned}$$

where $A_1B_1C_1$ is the orthic triangle of ABC. (This result was proved in chapter 5 on orthic triangles.)

If l is the common value of the sides of ABC then the orthic triangle $A_1B_1C_1$ is also equilateral and sidelengths are $\frac{l}{2}$. Thus

$$\begin{aligned} (\frac{|A_1B_1|+|B_1C_1|+|C_1A_1|}{3})^2 &= |A_1B_1|^2 \\ &= \frac{|A_1B_1|^2+|B_1C_1|^2+|C_1A_1|^2}{3}. \end{aligned}$$

The required result follows.