# 5. Orthic Triangle.

Let $ABC$ be a triangle with altitudes $AA_2, BB_2$ and $CC_2$. The altitudes are concurrent and meet at the orthocentre $H$ (Figure 1). The triangle formed by the feet of the altitudes, $A_2B_2C_2$ is the *orthic triangle*.

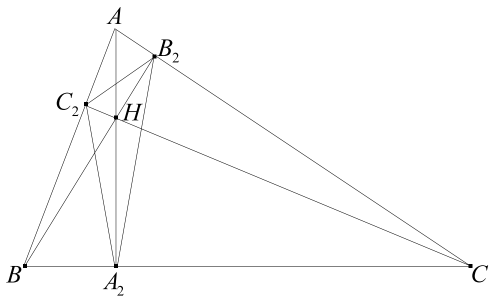


Figure 1:

Remarks There are several cyclic quadrilaterals :

- $AC_2HB_2, BC_2HA_2, CA_2HB_2$ are cyclic.
- $BCB_2C_2, ACA_2C_2, ABA_2B_2$ are cyclic.
- The sides of the orthic triangle are antiparallel with sides of the triangle $ABC$. We have $A_2B_2$ is antiparallel to $AB$, $B_2C_2$ is antiparallel to $BC$ and $C_2A_2$ is antiparallel to $CA$.

**Proposition 1** *If $ABC$ is an acute triangle, then the angles of the triangle $A_2B_2C_2$ are*

$$180^\circ - 2\widehat{A}, 180^\circ - 2\widehat{B} \text{ and } 180^\circ - 2\widehat{C}.$$

**Proof** Since $ACA_2C_2$ is cyclic, then

$$C_2\widehat{A_2}B = 180^\circ - C_2\widehat{A_2}C = \widehat{A}.$$

Since $ABA_2B_2$ is cyclic,

then $B_2\widehat{A_2}C = 180^\circ - B\widehat{A_2}B = \widehat{A}.$

Thus $C_2\widehat{A_2}B_2 = 180^\circ - 2\widehat{A}.$

Similarly, for the other two angles of $A_2B_2C_2$.

**Proposition 2** *The lengths of the sides of the orthic triangle are* $R\sin(2A) = a\cos(A), R\sin(2B) = b\cos(B)$ *and* $R\sin(2C) = c\cos(C)$*, where* $R$ *is the circumradius of the triangle* $ABC$*. Again,* $ABC$ *is an acute triangle.*

**Proof** Since the points $A_2B_2C_2$ lie on the ninepoint circle, the the circumcircle of $A_2B_2C_2$ has circumradius $R_{A_2B_2C_2}$ which is one half of $R$.

We now apply the sine rule to $A_2B_2C_2$. Then

$$\frac{|B_2C_2|}{\sin(\widehat{A_2})} = 2R_{A_2B_2C_2},$$

$$\begin{aligned}
\text{and so } \frac{|B_2C_2|}{\sin(180^\circ - 2A)} &= 2.\frac{R}{2}, \\
|B_2C_2| = R\sin(2\widehat{A}) &= 2R\sin(\widehat{A})\cos(\widehat{A}) \\
&= a\cos(\widehat{A}).
\end{aligned}$$

<u>Remark</u> In general, the side lengths of $A_2B_2C_2$ are

$$a|\cos(A)|, b|\cos(B)| \text{ and } c|\cos(C)|.$$

<u>Notation</u> If $ABC$ is a triangle, we denote the area of $ABC$ by $S(ABC)$.

**Proposition 3** *The area of* $A_2B_2C_2$ *is given by*

$$S(A_2B_2C_2) = \frac{R^2}{2}\sin(2\widehat{A})\sin(2\widehat{B})\sin(2\widehat{C}).$$

**Proof**

$$\begin{aligned}
\text{We have } S(A_2B_2C_2) &= \frac{|A_2C_2||A_2B_2|}{2}\sin(\widehat{A_2}) \\
&= \frac{R^2\sin(2\widehat{B})\sin(2\widehat{C})\sin(2\widehat{A})}{2} \\
&= \frac{R^2}{2}\sin(2\widehat{A})\sin(2\widehat{B})\sin(2\widehat{C}).
\end{aligned}$$

**Proposition 4** *Let* $r_{A_2B_2C_2}$ *and* $R_{A_2B_2C_2}$ *denote the inradius and circumradius of the orthic triangle* $A_2B_2C_2$*. Then*

$$r_{A_2B_2C_2} = 2R\cos(\widehat{A})\cos(\widehat{B})\cos(\widehat{C}) \quad \textit{and} \quad R_{A_2B_2C_2} = \frac{R}{2}.$$

**Proof** The value of $R_{A_2B_2C_2}$ follows from the fact that the ninepoint circle is the circumcircle of $A_2B_2C_2$ and its radius is one half of the circumradius of $ABC$.

For $r_{A_2B_2C_2}$ we have

$$
\begin{aligned}
r_{A_2B_2C_2} &= \frac{S(A_2B_2C_2)}{semiperimeter(A_2B_2C_2)} \\
&= \frac{(R^2/2)\sin(2\widehat{A})\sin(2\widehat{B})\sin(2\widehat{C})}{(R/2)(\sin(2\widehat{A})+\sin(2\widehat{B})+\sin(2\widehat{C}))} \\
&= R\,\frac{8\sin(\widehat{A})\sin(\widehat{B})\sin(\widehat{C})\cos(\widehat{A})\cos(\widehat{B})\cos(\widehat{C})}{4\sin(\widehat{A})\sin(\widehat{B})\sin(\widehat{C})} \\
&= 2R\cos(\widehat{A})\cos(\widehat{B})\cos(\widehat{C}).
\end{aligned}
$$

**Proposition 5** *If $A_2B_2C_2$ is the orthic triangle of a triangle $ABC$ and $H$ is the orthocentre of $ABC$ (Figure 2), then*

*(i) $H$ is the incentre of $A_2B_2C_2$, and*

*(ii) $A, B$ and $C$ are the centres of the excribed triangles.*

**Proof**

(i) Since $BA_2HC_2$ is cyclic,

$$C_2\widehat{A_2}H = C_2\widehat{B}H = A\widehat{B}H.$$

Since $CA_2HB_2$ is cyclic,

$$H\widehat{A_2}B_2 = H\widehat{C}B_2 = H\widehat{C}A.$$

Since $BCB_2C_2$ is cyclic,

$$C_2\widehat{B}B_2 = C_2\widehat{C}B_2$$

i.e. $$A\widehat{B}H = H\widehat{C}A.$$

Thus $$C_2\widehat{A_2}H = H\widehat{A_2}B_2$$

i.e. $AA_2$ is the bisector of the angle at $A_2$.

Similarly, $BB_2$ and $CC_2$ bisect the angles at $B_2$ and $C_2$. Thus the point $H$ is the incentre of the triangles $A_2B_2C_2$.

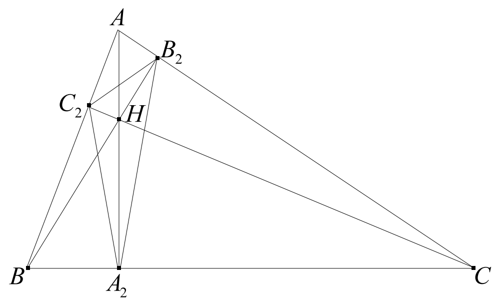

Figure 2:

(ii) Since $C_2A$ and $B_2A$ are perpendicular to the internal bisectors $C_2H$ and $B_2H$, then the point $A$ is where the external angle bisectors meet. Furthermore, $A$ lies on the internal bisector $HA_2$ of the angle at $A_2$. Thus $A$ is the centre of the escribed circle of $A_2B_2C_2$ which is externally tangent to the side $B_2C_2$. Similarly for the other two vertices $B$ and $C$.

**Theorem 1** *(Haghal)* *The perpendiculars from the vertices $A, B$ and $C$ to the sides $B_2C_2, C_2A_2$ and $A_2B_2$ are concurrent at the circumcentre $O$ of the triangle $ABC$.*

**Proof** Let $TA$ be tangent to the circumcircle of $ABC$ at the point $A$ (Figure 3).

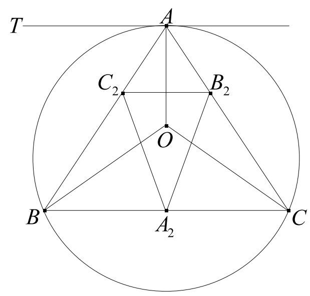


Figure 3:

We have that $B_2C_2$ is antiparallel to the side $BC$ and $AT$ is antiparallel to $BC$ (Step 1 of Feuerbach Theorem). Thus $TA$ is parallel to $BC$. If $O$ is the circumcentre of circumcircle of $ABC$, then $AT$ is perpendicular to $AO$. Thus $B_2C_2$ is perpendicular to $AO$. Similarly show that $BO$ is perpendicular to $A_2C_2$ and $CO$ is perpendicular to $A_2B_2$.

**Theorem 2** *Among all inscribed triangles in a triangle $ABC$, the perimeter is minimized by the orthic triangle.*

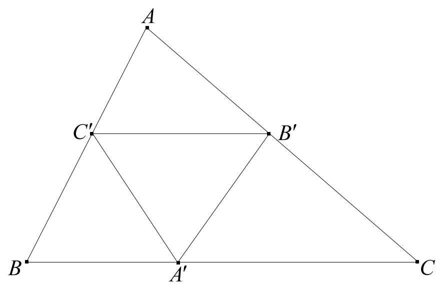


Figure 4:

**Proof** Let $A'B'C'$ be inscribed in the triangle $ABC$ (Figure 4).

Let $A''$ be the reflection of $A'$ in the side $AB$ and $A'''$ be the reflection of $A'$ on the side $AC$ (Figure 5).

$$\text{Then} \quad |C'A'| = |C'A''| \text{ and } |B'A'| = |B'A'''|$$

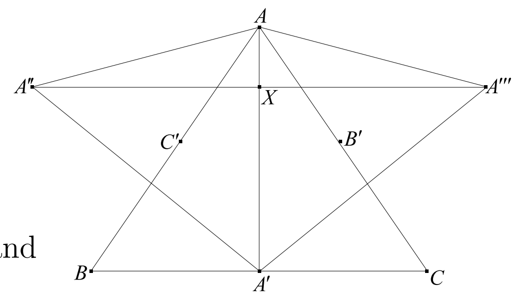


Figure 5:

Then if $\mathcal{P}$ denotes the perimeter, we have

$$\begin{aligned} \mathcal{P}(A'B'C') &= A'B' + B'C' + C'A' \\ &= |B'A'''| + |B'C'| + |C'A''| \\ &= |A'''B'| + |B'C'| + |C'A''| \\ &\geq |A'''A''|. \end{aligned}$$

Now consider the triangle $A''AA'''$. We have

$$\begin{aligned} & |AA''| = |AA'|, \\ & |AA'''| = |AA'|, \\ \text{so} \quad & |AA''| = |AA'''|. \end{aligned}$$

We also have $A''\widehat{A}B = A'\widehat{A}B$ and $A'\widehat{A}C = A'''\widehat{A}C$. Thus $A''\widehat{A}A''' = 2\widehat{A}$.

Let $\gamma$ be the angle $A\widehat{A''}A''' = A\widehat{A'''}A''$. If $X$ is the point of point of intersection of the lines $AA'$ and $A''A'''$, then

$$\begin{aligned} \frac{|A''X|}{|A''A|} &= \cos(\gamma). \\ \text{Thus} \quad |A''A'''| &= 2|A''X| \\ &= 2\cos(\gamma)|A''A| \\ &= 2\sin(\widehat{A})|A''A|, \end{aligned}$$

since $180° = 2\gamma + 2\widehat{A}$ so $\gamma + \widehat{A} = 90°$ and thus $\cos(\gamma) = \cos(90° - \widehat{A}) = \sin(\widehat{A})$.

$$\text{But } |A''A| = |AA'| \geq |AA_2|, \text{ so}$$
$$|A''A'''| \geq 2\sin(\widehat{A})|AA_3|.$$

Thus, if $A'B'C'$ is an inscribed triangle, with $B'$ and $C'$ fixed, perimeter is minimised if $A$ is the point $A_2$. Similarly the perimeter is further mimimised by taking $B'$ and $C'$ to be the points $B_2$ and $C_2$ respectively. Result follows.

**Theorem 3** *If $ABC$ is an acute triangle which is not isosceles and $A_2B_2C_2$ is the orthic triangle then the points $A', B'$ and $C'$, where the sides $B_2C_2$ and $BC$ intersect, $A_2C_2$ and $AC$ intersect and $A_2B_2$ and $AB$ intersect, respectively, are collinear (Figure 6).*

<u>Remark</u> The line containing these points is called the *orthic line* of the triangle $ABC$.

**Proof** If we are given two non-concentric circles then the locus of points whose powers with respect to the circles is a line perpendicular to the line joining the centres of the circles. It is called the radical axis of the circles.

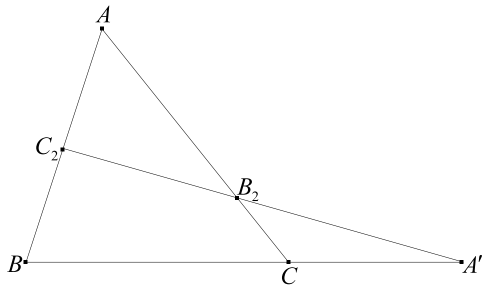


Figure 6:

Since $BC_2B_2C$ is cyclic, then
$|A'B_2|.|A'C_2| = |A'C||A'B|$.

If $\mathcal{C}$ is the circumcircle of $ABC$ and $\mathcal{C}_9$ is the ninepoint circle, thus

$$\rho_{\mathcal{C}}(A') = \rho_{\mathcal{C}_9}(A').$$

Similarly, $\rho_{\mathcal{C}}(B') = \rho_{\mathcal{C}_9}(B')$ and $\rho_{\mathcal{C}}(C') = \rho_{\mathcal{C}_9}(C')$.

Thus the 3 points $A', B'$ and $C'$ lie on the radical axis of the circles $\mathcal{C}$ and $\mathcal{C}_9$.