Chapter 4. Feuerbach's Theorem

Let A be a point in the plane and k a positive number. Then in the previous chapter we proved that the inversion mapping with centre A and radius k is the mapping

$$Inv : \mathcal{P}\backslash\{A\} \to \mathcal{P}\backslash\{A\}$$

which is defined as follows. If B_1 is a point, then $Inv(B_1) = B_2$ if B_2 lies on the line joining A and B_1 and

$$|AB_1||AB_2| = k^2.$$

We denote this mapping by $Inv(A, k^2)$. We proved the following four properties of the mapping $Inv(A, k^2)$.

(a) If A belongs to a circle $\mathcal{C}(O, r)$ with centre O and radius r, then $Inv(\mathcal{C}(O, r))$ is a line l which is perpendicular to OA.

(b) If l is a line which does not pass through A, then $Inv(l)$ is a circle such that l is perpendicular to the line joining A to the centre of the circle.

(c) If A does not belong to a circle $\mathcal{C}(O, r)$, then

$$Inv(\mathcal{C}(O, r)) = \mathcal{C}(O, r)$$
$$\text{with } r' = r.\frac{k^2}{\rho(A, \mathcal{C}(O, r))}$$

(d) If $Inv(B_1) = B_2$ and $Inv(C_1) = C_2$, where B_1 and C_1 are two points in the plane, then

$$|B_2C_2| = |B_1C_1|.\frac{k^2}{|AB_1||AC_1|}.$$

Remark Let A be an arbitrary point which does not belong to the circle $\mathcal{C}(O,r)$ with centre O and radius r and let $\rho_{\mathcal{C}}(A)$ be the power of A with respect to the circle $\mathcal{C} = \mathcal{C}(O,r)$. Then if Inv is the mapping with pole (centre) A and $k^2 = \rho_{\mathcal{C}}(A)$, i.e.

$$Inv := Inv(A, \rho_{\mathcal{C}}(A))$$

then $Inv(\mathcal{C}(O,r)) = \mathcal{C}(O,r)$, i.e. $\mathcal{C}(O,r)$ is invariant under the mapping Inv. This follows from the following observations. Since A does not belong to $\mathcal{C}(O,r)$, then $Inv(\mathcal{C}(O,r))$ is a circle with radius r' where

$$\begin{aligned} r' &= r\frac{k^2}{\rho_{\mathcal{C}}(A)}, \text{by (c) above} \\ &= r, \text{ since } k^2 = \rho_{\mathcal{C}}(A) \end{aligned}$$

Furthermore, if P is any point of $\mathcal{C}(O,r)$ and $P' = Inv(P)$, then

$$|AP||AP'| = \rho_{\mathcal{C}}(A).$$

Thus P' is also on the circle $\mathcal{C}(O,r)$, so the result follows.

Feuerbach's Theorem *The nine point circle of a triangle is tangent to the incircle and the three excircles of the triangle.*

We prove this using inversion. The proof is developed through a sequence of steps.

Step 1 Let ABC be a triangle and let Inv be the mapping $Inv(A, k^2)$ for some $k > 0$. If $\mathcal{C}(O,R)$ denotes the circumcircle of ABC, then

$$Inv(\mathcal{C}(ABC))$$

is a line L which is antiparallel to the line BC.

Proof Let O be the circumcentre of the triangle ABC and let Inv denote the mapping $Inv(A, k^2)$. From part (a) of the proposition listing the properties of inversion maps, $Inv(\mathcal{C}(ABC)) = B_1C_1$ where B_1, C_1 are images of B and C under Inv. Then the line through B_1 and C_1 is perpendicular to line AO (Figure 1).

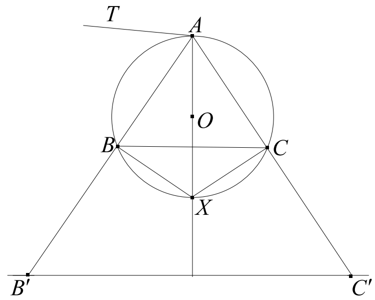

Figure 1:

Now let TA be tangent to $\mathcal{C}(ABC)$ at A. Then if X is the point of intersection of AO with $\mathcal{C}(ABC)$, we have

$$\begin{aligned} T\widehat{A}B &= 90^\circ - B\widehat{A}X \\ &= B\widehat{X}A \\ &= B\widehat{C}A. \end{aligned}$$

Since $TA \perp AO$ and $AO \perp B'C'$, then $TA \| B'C'$ so

$$T\widehat{A}B = A\widehat{B'}C'.$$

Thus $A\widehat{B'}C' = B\widehat{C}A$ and so $B'C'$ is antiparallel to BC, as desired.

Step 2 Let ABC be a triangle. The incircle $\mathcal{C}(I, r)$, with centre I and radius r, touches the sides BC, CA and AB at the points P, Q and R respectively (Figure 2). If $s = \frac{1}{2}(a + b + c)$ denotes the semiperimeter, we have

$$\begin{aligned} |CP| &= |CQ| = s - c, \\ |BP| &= |BR| = s - b, \\ |AR| &= |AQ| = s - a. \end{aligned}$$

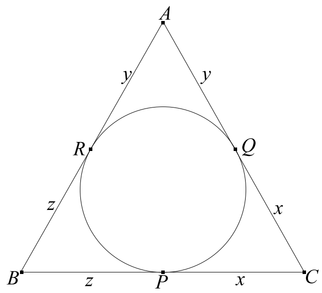

Figure 2:

Proof Let

$$\begin{aligned} x &= |CP| = |CQ|, \\ y &= |AR| = |AQ|, \\ x &= |BR| = |BP|. \end{aligned}$$

Then $s = x + y + z$ and $a = x + z; b = x + y$ and $c = y + z$.

So $|CP| = |CQ| = x = (x + y + z) - (y + z) = s - c$.
Similarly for the other lengths.

Step 3 Let ABC be a triangle and let $\mathcal{C}(I_a, r_a)$ be the excircle touching the side BC and the sides AB and AC externally at the points P_a, R_a and Q_a respectively (Figure 3). Then

$$|BP_a| = s - c \text{ and } |CP_a| = s - b.$$

Proof Let

$$\begin{aligned} |AR_a| &= |AQ_a| = x, \\ |CQ_a| &= |CP_a| = y, \\ |BP_a| &= |BR_a| = z. \end{aligned}$$

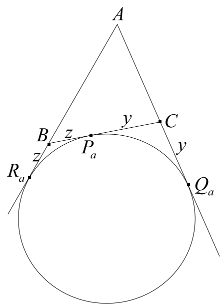

Figure 3:

Then

$$\begin{aligned} x - y &= b, \\ x - z &= c, \\ y + z &= a. \end{aligned}$$

Adding, we get $2x = a + b + c$ so $x = s$.

From this $y = x - b = s - b$,
so $|CP_a| = |CQ_a| = s - b$,
and $z = x - c = s - c$,
so $|BP_a| = |BR_a| = s - c$, as required.

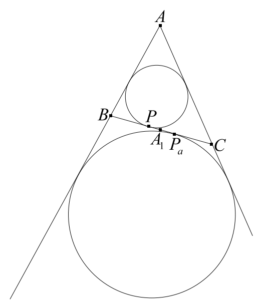

Figure 4:

Remark If A_1 is the midpoint of the side BC, then

$$|A_1P| = |A_1P_a| \text{ (Figure 4).}$$

This follows from the observation that

$$|BP| = s - b \text{ (Step 2)}$$
$$|CP_a| = s - b \text{ (Step 3)}$$

Then $|A_1P| = \dfrac{a}{2} - (s-b) = \dfrac{b-c}{2}$,

$|A_1P_a| = \dfrac{a}{2} - (s-b) = \dfrac{b-c}{2}$.

Step 4 If ABC is a triangle and A_3 is a point on the side BC where the bisector of the angle at A meets BC (Figure 5), then

$$|BA_3| = \frac{ac}{b+c} \text{ and } |CA_3| = \frac{ab}{b+c}.$$

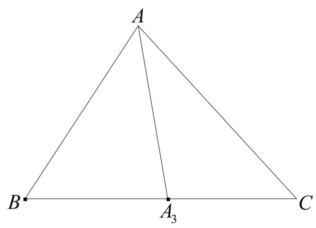

Figure 5:

Proof $\dfrac{|BA_3|}{|CA_3|} = \dfrac{area(ABA_3)}{area(ACA_3)} = \dfrac{|AB||AA_3|\sin(\frac{\widehat{A}}{2})}{|AC||AA_3|\sin(\frac{\widehat{A}}{2})} =$

$\dfrac{|AB|}{|AC|} = \dfrac{a}{b}$.

Since $|BA_3| + |CA_3| = a$, then $\dfrac{|BA_3|}{a - |BA_3|} = \dfrac{c}{b}$.

Solve for $|BA_3|$ to get $|BA_3| = \dfrac{ac}{b+c}$.

Finally, $|CA_3| = a - \dfrac{ac}{b+c} = \dfrac{ab}{b+c}$.

Step 5 In a triangle ABC let A_1 be the midpoint of the side BC and let A_2 be the foot of the perpendicular from A to BC (Figure 6). Then

$$|A_1A_2| = \frac{b^2 - c^2}{2a}.$$

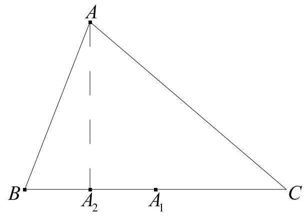

Figure 6:

Proof We have

$$\begin{aligned} b^2 - c^2 &= |AA_2|^2 + |A_2C|^2 - |AA_2|^2 - |A_2B|^2 \\ &= (|A_2C| + |A_2B|)(|A_2C| - |A_2B|) \\ &= a\{|A_1A_2| + |A_1C| - |A_1B| + |A_1A_2|\} \\ &= 2a|A_1A_2|. \end{aligned}$$

Thus $|A_1A_2| = \dfrac{b^2 - c^2}{2a}$, as required.

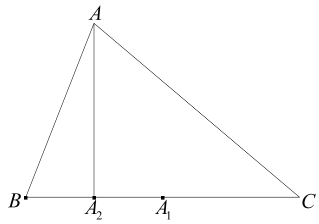

Figure 7:

Step 6 Let $\mathcal{C}(O, r)$ be a circle with centre O and radius r and let A be an arbitrary point not belonging to $\mathcal{C}(O, r)$. Consider the inversion with pole A and $k^2 = \rho_{\mathcal{C}}(A)$, the power of A with respect to the circle $\mathcal{C}(O, r)$. Then the circle $\mathcal{C}(O, r)$ remains invariant under the inversion $Inv(A, \rho_{\mathcal{C}}(A))$.

Proof Denote by Inv the inversion $Inv(A, \rho_{\mathcal{C}}(A))$. Since A does not belong to $\mathcal{C}(O, r)$, then

$$Inv(\mathcal{C}(O, r)) \text{ is a circle with radius } r' \text{ where}$$
$$r' = r.\frac{k^2}{\rho_{\mathcal{C}}} = r,$$

since we have $k^2 = \rho_{\mathcal{C}}(A)$.

Now choose a point B on $\mathcal{C}(O, r)$ and let $B' = Inv(B)$. Then $|AB||AB'| = k^2 = \rho_{\mathcal{C}}(A)$. But this implies that B' is a point of $\mathcal{C}(O, r)$. Thus

$$Inv(\mathcal{C}(O, r)) = \mathcal{C}(O, r),$$

as required.

Step 7 In a triangle ABC let A_1, A_2, A_3 and P be the following points on the side BC;
A_1 is the midpoint,
A_2 is the foot of the altitude from A,
A_3 is the point where the bisector of $\widehat{A}$ meets BC,
P is the foot of the perpendicular from the incentre I to BC and so is the point of tangency of BC with the incircle (Figure 8).

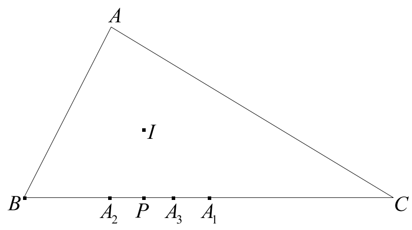

Figure 8:

$$\text{Then } |A_1P|^2 = |A_1A_2||A_1A_3|$$

Proof

$$\begin{aligned} \text{We have} \quad |A_1P| &= |A_1B| - |BP| \\ &= \frac{a}{2} - (s-b) \qquad \text{(step 3)} \\ &= \frac{b-c}{2} \\ |A_1A_2| &= \frac{b^2-c^2}{2a} \qquad \text{(step 5)} \\ |A_1A_3| &= |BA_1| - |BA_3| \\ &= \frac{a}{2} - \frac{ac}{b+c} \qquad \text{(step 4)} \\ &= \frac{a(b-c)}{2(b+c)}. \end{aligned}$$

It follows that

$$|A_1P|^2 = |A_1A_3||A_1A_2| = (\frac{b-c}{2})^2,$$

as required.

We now return to the proof of Feuerbach's theorem which states that the nine point circle $\mathcal{C}_9$ of a triangle ABC is tangent to the incircle and the three escribed circles of the triangle.

Let A_1 be the midpoint of the side BC and let P and P_a be the points of tangency of the incircle and the escribed circle drawn external to side BC, respectively (Figure 9). We consider the inversion mapping $Inv(A_1, k^2)$ where $k^2 = |A_1P|^2$ and we denote it by Inv.

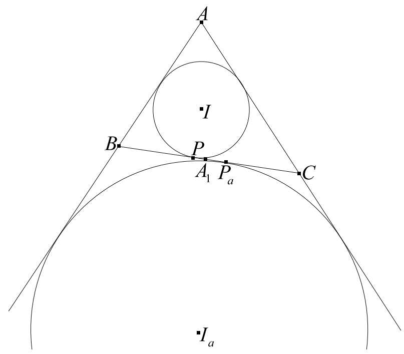

Figure 9:

Since P is the point of tangency of the side BC with the incircle $\mathcal{C}(I, r)$, then

$$|A_1P|^2 = \rho_{\mathcal{C}(I,r)}(A_1)$$

By step 6, it follows that

$$Inv(\mathcal{C}(I, r)) = \mathcal{C}(I, r)$$

Since P_a is the point of tangency of the side BC with the escribed circle $\mathcal{C}(I_a, r_a)$ and $|A_1P_a| = |A_1P|$, then $\rho_{\mathcal{C}(I_a,r_a)}(A_1) = |A_1P_a|^2 = |A_1P|^2 = k^2$, then

$$Inv(\mathcal{C}(I_a, r_a)) = \mathcal{C}(I_a, r_a).$$

Thus $\mathcal{C}(I, r)$ and $\mathcal{C}(I_a, r_a)$ are both invariant under the mapping Inv. Now we consider the image of the nine-point circle under Inv. Since A_1 belongs to the nine-point circle $\mathcal{C}_9$ and A_1 is the pole of Inv, then $Inv(\mathcal{C}_9)$ is a line d. But $\mathcal{C}_9$ is the circumcircle of the triangle with vertices the midpoints A_1, B_1 and C_1 of the sides of the triangle ABC so the line d is antiparallel to the line B_1C_1 (step 1). Since $B_1C_1 \| BC$ then d is antiparallel to the side BC.

We also have that $|A_1A_2||A_1A_3| = |A_1P|^2$ (step 7) and since A_2 belongs to $\mathcal{C}_9$ then d is a line which passes through A_3, as $Inv(A_2) = A_3$.

Now let $B'C'$ be the second common tangent of the two circles $\mathcal{C}(I, r)$ and $\mathcal{C}(I_a, r_a)$. Since A_3 is the bisector of the angle at A, these common tangents intersect at A_3 (Figure 10). Now claim that $A\widehat{B}C = A\widehat{C'}B'$ and $A\widehat{B'}C' = A\widehat{C}B$. From this it follows that the second common tangent $B'C'$ is antiparallel to the side BC. Since A_3

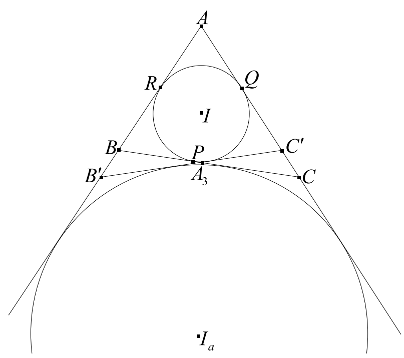

Figure 10:

is on $B'C'$ then it follows that the line d must be $B'C'$, that is

$$Inv(\mathcal{C}_9) = \text{line } d.$$

Finally, since d is tangent to $\mathcal{C}(I, r)$ and $\mathcal{C}(I_a, r_a)$, then $\mathcal{C}_9 = (Inv)^{-1}(d)$ is tangent to $\mathcal{C}(I, r)$ and $\mathcal{C}(I_a, r_a)$. Thus $\mathcal{C}_9$ is tangent to the incircle and escribed circle external to the side BC. Similarly it can be shown that $\mathcal{C}_9$ is also tangent to the other two escribed circles.

It remains to show that the common tangents BC and $B'C'$ are antiparallel.

Let P, Q and R be the points of tangency of the sides BC, CA and AB with the incircle $\mathcal{C}(I, r)$ of the triangle ABC. Let P' be the point of tangency of the second common tangent $B'C'$ with the incircle $\mathcal{C}(I, r)$ (Figure 11).

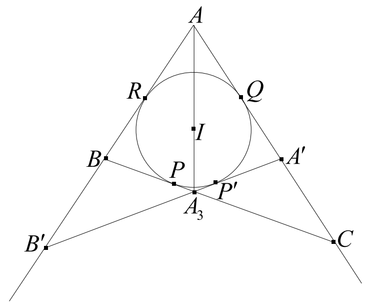

Figure 11:

The triangles AIR and AIQ are similar so $A\widehat{I}R = A\widehat{I}Q$.

The triangles A_3IP and A_3IP' are similar so $A_3\widehat{I}P = A_3\widehat{I}P'$.

$$\begin{aligned} \text{Then } P\widehat{I}R &= 180^\circ - (A\widehat{I}R + A_3\widehat{I}P) \\ &= 180^\circ - (A\widehat{I}Q + A_3\widehat{I}P') \\ &= P'\widehat{I}Q. \end{aligned}$$

Since the quadrilaterals $PIRB$ and $P'IQC'$ are cyclic, then

$$\begin{aligned} P\widehat{B}R &= 180^\circ - P\widehat{I}R \\ &= 180^\circ - P'\widehat{I}Q \\ &= P'\widehat{C'}Q, \\ \text{i.e., } A\widehat{B}C &= A\widehat{C'}B'. \end{aligned}$$

$$\begin{aligned} \text{Thus } B\widehat{C}A &= 180^\circ - (\widehat{A} + A\widehat{B}C) \\ &= 180^\circ - (\widehat{A} + A\widehat{C'}B') \\ &= A\widehat{B'}C'. \end{aligned}$$

It follows that BC and $B'C'$ are antiparallel, as required.

The point P', where the second tangent $B'C'$ touches $\mathcal{C}(I, r)$ is called the *Feuerbach point.*