Chapter 1. The Medial Triangle

The triangle formed by joining the midpoints of the sides of a given triangle is called the medial triangle. Let $A_1B_1C_1$ be the medial triangle of the triangle ABC in Figure 1. The sides of $A_1B_1C_1$ are parallel to the sides of ABC and half the lengths. So $A_1B_1C_1$ is $\frac{1}{4}$ the area of ABC.

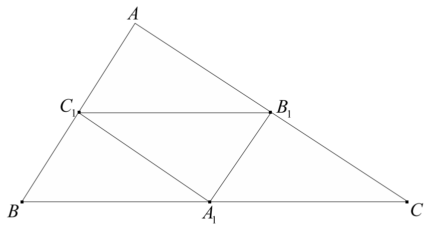


Figure 1:

In fact

$$\begin{aligned} area(AC_1B_1) &= area(A_1B_1C_1) &= area(C_1BA_1) \\ &= area(B_1A_1C) &= \frac{1}{4}\, area(ABC). \end{aligned}$$

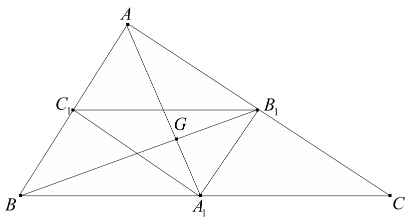


Figure 2:

The quadrilaterals $AC_1A_1B_1$ and $C_1BA_1B_1$ are parallelograms. Thus the line segments AA_1 and C_1B_1 bisect one another, and the line segments BB_1 and CA_1 bisect one another. (Figure 2)

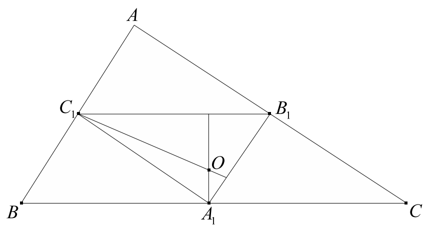


Figure 3:

Thus the medians of $A_1B_1C_1$ lie along the medians of ABC, so both triangles $A_1B_1C_1$ and ABC have the same centroid G.

Now draw the altitudes of $A_1B_1C_1$ from vertices A_1 and C_1. (Figure 3) These altitudes are perpendicular bisectors of the sides BC and AB of the triangle ABC so they intersect at O, the circumcentre of ABC. Thus the orthocentre of $A_1B_1C_1$ coincides with the circumcentre of ABC.

Let H be the orthocentre of the triangle ABC, that is the point of intersection of the altitudes of ABC. Two of these altitudes AA_2 and BB_2 are shown. (Figure 4) Since O is the orthocentre of $A_1B_1C_1$ and H is the orthocentre of ABC then

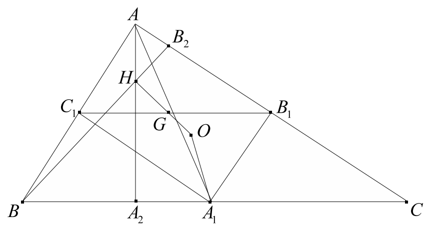


Figure 4:

$$|AH| = 2|A_1O|$$

.

The centroid G of ABC lies on AA_1 and

$$|AG| = 2|GA_1|$$

. We also have $AA_2 \| OA_1$, since O is the orthocentre of $A_1B_1C_1$. Thus

$$H\widehat{A}G = G\widehat{A_1}O,$$

and so triangles HAG and GA_1O are similar.

$$\begin{gathered}\text{Since } H\widehat{A}G = G\widehat{A}_1O,\\ |AH| = 2|A_1O|,\\ |AG| = 2|GA_1|.\end{gathered}$$

Thus

$$A\widehat{G}H = A_1\widehat{G}O,$$

i.e. H, G and O are collinear. Furthermore, $|HG| = 2|GO|$.
Thus

Theorem 1 *The orthocentre, centroid and circumcentre of any triangle are collinear. The centroid divides the distance from the orthocentre to the circumcentre in the ratio* $2:1$.

The line on which these 3 points lie is called the **Euler line** of the triangle.

We now investigate the circumcircle of the medial triangle $A_1B_1C_1$. First we adopt the notation

$$\mathcal{C}(ABC)$$

to denote the circumcircle of the triangle ABC.

Let AA_2 be the altitude of ABC from the vertex A. (Figure 5) Then

$$A_1B_1 \| AB \text{ and}$$
$$|A_1B_1| = \frac{1}{2}|AB|.$$

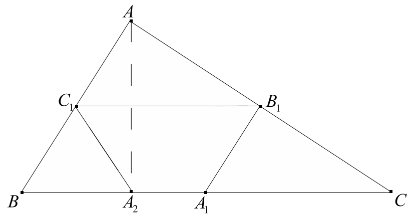


Figure 5:

In the triangle AA_2B, $A\widehat{A_2}B = 90°$ and C_1 is the midpoint of AB. Thus

$$|A_2C_1| = \frac{1}{2}|AB|.$$

Thus $C_1B_1A_1A_2$ is an isoceles trapezoid and thus a cyclic quadrilateral. It follows that A_2 lies on the circumcircle of $A_1B_1C_1$. Similarly for the points B_2 and C_2 which are the feet of the altitudes from the vertices B and C.

Thus we have

Theorem 2 *The feet of the altitudes of a triangle ABC lie on the circumcircle of the medial triangle $A_1B_1C_1$.*

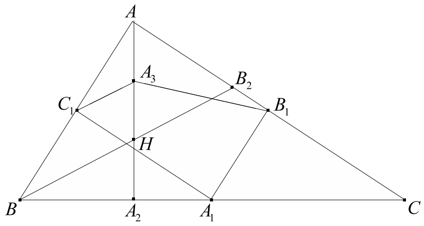


Figure 6:

Let A_3 be the midpoint of the line segment AH joining the vertex A to the orthocentre H. (Figure 6) Then we claim that A_3 belongs to $\mathcal{C}(A_1B_1C_1)$, or equivalently $A_1B_1A_3C_1$ is a cyclic quadrilateral.

We have $C_1A_1 \| AC$ and $C_1A_3 \| BH$, but $BH \perp AC$.
Thus $C_1A_1 \perp C_1A_3$.
Furthermore $A_3B_1 \| HC$ and $A_1B_1 \| AB$.
But $HC \| AB$, thus $A_3B_1 \perp B_1A_1$.

Thus, quadrilateral $C_1A_1B_1A_3$ is cyclic, i.e. $A_3 \in \mathcal{C}(A_1B_1C_1)$.
Similarly, if B_3 C_3 are the midpoints of the line segments HB and HC respectively then B_3, $C_3 \in \mathcal{C}(A_1B_1C_1)$.

Thus we have

Theorem 3 *The 3 midpoints of the line segments joining the orthocentre of a triangle to its vertices all lie on the circumcircle of the medial triangle.*

Thus we have the 9 points $A_1, B_1, C_1, A_2, B_2, C_2$ and A_3, B_3, C_3 concyclic. This circle is the **ninepoint circle** of the triangle ABC.

Since $C_1A_1B_1A_3$ is cyclic with $A_3C_1 \perp C_1A_1$ and $A_3B_1 \perp B_1A_1$, then A_1A_3 is a diameter of the ninepoint circle. Thus the centre N of the ninepoint circle is the midpoint of the diameter A_1A_3. We will show in a little while that N also lies on the Euler line and that it is the midpoint of the line segment HO joining the orthocentre H to the circumcentre O.

Definition 1 *A point A' is the* symmetric point *of a point A through a third point O if O is the midpoint of the line segment AA'. (Figure 7)*

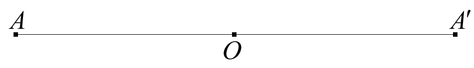


Figure 7:

We now prove a result about points lying on the circumcircle of a triangle.

Let A' be the symmetric point of H through the point A_1 which is the midpoint of the side BC of a triangle ABC. Then we claim that A' belongs to $\mathcal{C}(ABC)$. To see this, proceed as follows.

The point A_1 is the midpoint of the segments HA' and BC so $HBA'C$ is a parallelogram. Thus $A'C \| BH$. But BH extended is perpendicular to AC

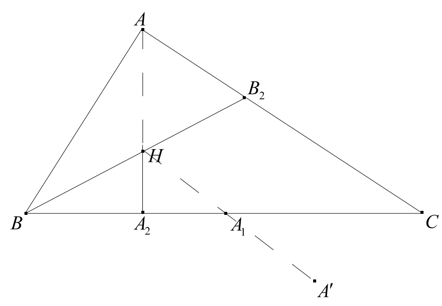


Figure 8:

and so $A'C \perp AC$. Similarly $BA' \| CH$ and CH is perpendicular to AB so $BA' \perp AB$. Thus

$$A\widehat{B}A' = A'\widehat{C}A = 90°.$$

Thus $ABA'C$ is cyclic and furthermore AA' is a diameter. Thus

$$A' \in \mathcal{C}(ABC).$$

Similarly the other symmetric points of H through B_1 and C_1, which we denote by B' and C' respectively, also lie on $\mathcal{C}(ABC)$.

Now consider the triangle AHA'. The points A_3, A_1 and O are the midpoints of the sides AH, HA' and $A'A$ respectively. Thus HA_1OA_3 is a parallelogram so HO bisects A_3A_1. We saw earlier that the segment A_3A_1 is a diameter of $\mathcal{C}(A_1B_1C_1)$ so the midpoint of HO is also then the centre of $\mathcal{C}(A_1B_1C_1)$. Thus the centre N of the ninepoint circle, i.e. $\mathcal{C}(A_1B_1C_1)$ lies on the Euler line and is the midpoint of the segment HO.

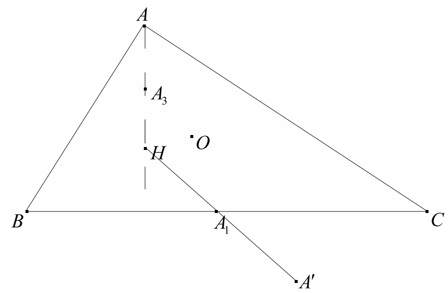


Figure 9:

Furthermore the radius of the ninepoint circle is one half of the radius R of the circumcircle $\mathcal{C}(ABC)$.

Now consider the symmetric points of H through the points A_2, B_2 and C_2 where again these are the feet of the perpendiculars from the vertices. We claim these are also on $\mathcal{C}(ABC)$.

Let BB_2 and CC_2 be altitudes as shown in Figure 10, H is the point of intersection.

Then AC_2HB_2 is cyclic so $C_2\widehat{A}B_2 + C_2\widehat{H}B_2 = 180°$.

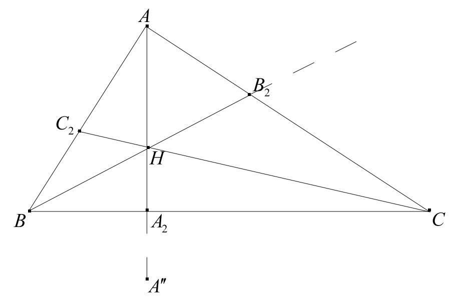


Figure 10:

But $BHC = C_2\widehat{H}B_2$ so

$$\begin{aligned} BHC &= 180^\circ - C_2\widehat{A}B_2 && \text{which we write as} \\ &= 180^\circ - \widehat{A} \end{aligned}$$

By construction, the triangles BHC and $BA''C$ are congruent, so $B\widehat{H}C = BA''C$. Thus

$$BA''C = 180^\circ - \widehat{A},$$

and so $ABA''C$ is cyclic. Thus

$$A'' \in \mathcal{C}(ABC)$$

Similarly for B'' and C''.

Returning to the symmetric points through A_1, B_1 and C_1 of the orthocentre, we can supply another proof of Theorem 1.

Theorem 1 *The orthocentre H, centroid G and circumcentre O of a triangle are collinear points.*

Proof In the triangle AHA', the points O and A_1 are midpoints of sides AA' and HA' respectively. (Figure 11) Then the line segments AA_1 and HO are medians, which intersect at the centroid G' of $\triangle AHA'$ and furthermore

$$\frac{|G'H|}{|G'O|} = 2 = \frac{|G'A|}{|G'A_1|}$$

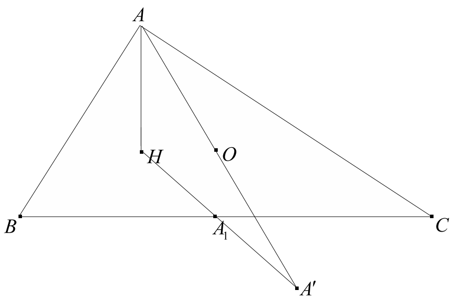


Figure 11:

But AA_1 is also a median of the triangle ABC so the centroid G lies on AA_1 with

$$\frac{|GA|}{|GA_1|} = 2$$

Thus G' coincides with G and so G lies on the line OH with

$$\frac{|GH|}{|GO|} = 2$$

<u>Remark</u> On the Euler line the points H (orthocentre), N (centre of ninepoint circle), G (centroid) and O (circumcircle) are located as follows:

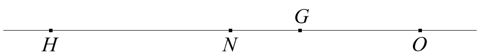


Figure 12:

$$\text{with } \frac{|HN|}{|NO|} = 1 \qquad \text{and } \frac{|HG|}{|GO|} = 2$$

...

If $ABCD$ is a cyclic quadrilateral, the four triangles formed by selecting 3 of the vertices are called the diagonal triangles. The centre of the circumcircle of $ABCD$ is also the circumcentre of each of the diagonal triangles.

We adopt the notational convention of denoting points associated with each of these triangles by using as subscript the vertex of the quadrilateral which is not a vertex of the diagonal triangle. Thus H_A, G_A and I_A will denote the orthocentre, the centroid and the incentre respectively of the triangle BCD. (Figure 13)

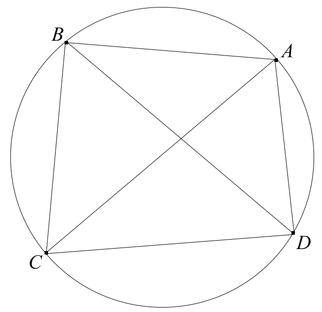


Figure 13:

Our first result is about the quadrilateral formed by the four orthocentres.

Theorem 4 *Let $ABCD$ be a cyclic quadrilateral and let H_A, H_B, H_C and H_D denote the orthocentres of the diagonal triangles BCD, CDA, DAB and ABC respectively. Then the quadrilateral*

$$H_A H_B H_C H_D$$

is cyclic. It is also congruent to the quadrilateral $ABCD$.

Proof Let M be the midpoint of CD and let A' and B' denote the symmetric points through M of the orthocentres H_A and H_B respectively. (Figure 14)

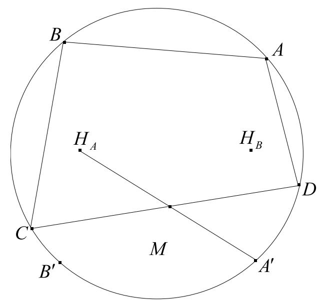


Figure 14:

The lines AA' and BB' are diagonals of the circumcircleof $ABCD$ so $ABB'A'$ is a rectangle. Thus the sides AB and $A'B'$ are parallel and of same length.

The lines H_AA' and H_BB' bisect one another so $H_AH_BA'B'$ is a parallelogram so we get that H_AH_B is parallel to $A'B'$ and are of the same length. Thus

$$H_AH_B \text{ and } AB \text{ are parallel and are of the same length.}$$

Similarly one shows that the remaining three sides of $H_AH_BH_CH_O$ are parallel and of the same length of the remaining 3 sides of $ABCD$. The result then follows.

The next result is useful in showing that in a cyclic quadrilateral, various sets of 4 points associated with the diagonal triangles form cyclic quadrilaterals.

Proposition 1 *Let $ABCD$ be a cyclic quadrilateral and let $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ and $\mathcal{C}_4$ be circles through the pair of points*

$$A, B; \quad B, C; \quad C, D; \quad D, A$$

and which intersect at points A_1, B_1, C_1 and D_1. (Figure 15) Then $A_1B_1C_1D_1$ is cyclic.

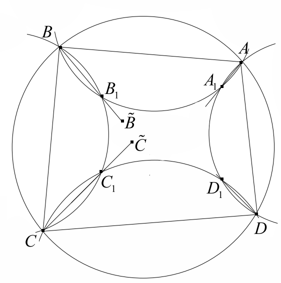


Figure 15:

Proof Let B_1 be the point where circles through AB and BC meet. Similarly for the points A_1, C_1 and D_1. Join the points AA_1, BB_1, CC_1 and DD_1, extend to points $\tilde{A}, \tilde{B}, \tilde{C}$ and $\tilde{D}$. (These extensions are for convenience of referring to angles later.)

We will apply to the previous diagram the result that if we have a cyclic quadrilateral then an exterior angle is equal to the opposite angle of the quadrilateral. In Figure 16, if CB is extended to $\tilde{B}$ then:

$$\begin{aligned} \tilde{B}\widehat{B}A + A\widehat{B}C &= 180^\circ \\ C\widehat{D}A + A\widehat{B}C &= 180^\circ \\ \text{Thus} \quad \tilde{B}\widehat{B}C &= C\tilde{D}A \\ &\ldots \end{aligned}$$

In Figure 17 we apply the above result to 4 quadrilaterals.

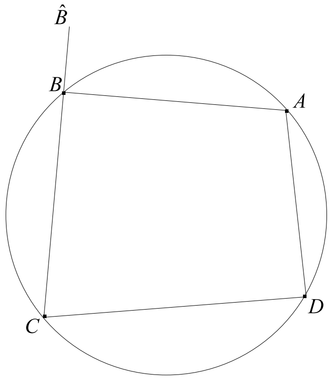


Figure 16:

In C_1CDD_1

$$\tilde{C}\widehat{C_1}D_1 = C\widehat{D}D_1$$
$$\tilde{D}\widehat{D_1}C_1 = C_1\widehat{C}D$$

In BCC_1B_1

$$\tilde{C}\widehat{C_1}B_1 = C\widehat{B}B_1$$
$$\tilde{D}\widehat{B_1}C_1 = C_1\widehat{C}B$$

In BB_1A_1A

$$\tilde{B}\widehat{B}_1A_1 = B\widehat{A}A_1$$
$$\tilde{A}A_1B_1 = A\widehat{B}B_1$$

In AA_1D_1D

$$\tilde{A}A_1D_1 = A\widehat{D}D_1$$
$$\tilde{D}D_1A_1 = D\widehat{A}A_1$$

Adding up angles

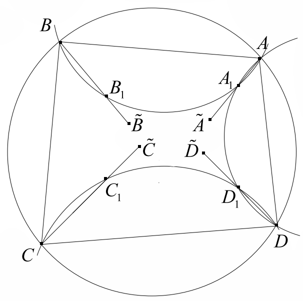


Figure 17:

$$\begin{aligned} C_1\widehat{D}_1A_1 + C_1\widehat{B}_1A_1 &= C_1\widehat{D}_1\tilde{D} + \tilde{D}\widehat{D_1}A_1 + C_1\widehat{B_1}\tilde{B} + \tilde{B}\widehat{B_1}A_1 \\ &= C_1\widehat{C}D + A_1\widehat{A}D + C_1\widehat{C}B + B\widehat{A}A_1 \\ &= C_1\widehat{C}D + C_1\widehat{C}B + A_1\widehat{A}D + B\widehat{A}A_1 \\ &= B\widehat{C}D + B\widehat{A}D + 180^\circ \end{aligned}$$

Similarly

$$D_1\widehat{C}_1B_1 + D_1\widehat{A}_1B_1 = 180^\circ$$

Thus $A_1B_1C_1D_1$ is cyclic. $\square$

We now apply this to orthocentres and incentres of the diagonal triangles.

Theorem 5 *Let $ABCD$ be a cyclic quadrilateral and let H_A, H_B, H_C and H_D be the orthocentres of the diagonal triangles BCD, CDA, DAB and ABC, respectively. Then $H_AH_BH_CH_D$ is a cyclic quadrilateral.*

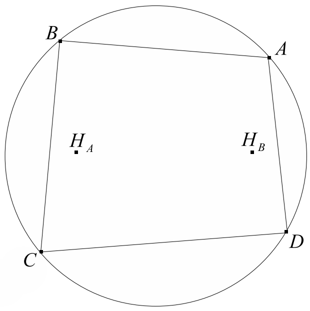


Figure 18:

Proof In the triangle BCD, (Figure 18),

$$C\widehat{H_A}D = 180^\circ - C\widehat{B}D.$$

In the triangle CDA, $C\widehat{H_B}D = 180^\circ - D\widehat{B}C$
But

$$\begin{aligned} D\widehat{A}C &= D\widehat{B}C \text{ so} \\ C\widehat{H_A}D &= C\widehat{H_B}D \end{aligned}$$

and so we conclude that CDH_BH_A is cyclic. Similarly, show that H_BDAH_C, H_CABH_D and H_DBCH_A are cyclic.

Now apply the proposition to see that $H_AH_BH_CH_D$ is cyclic.

Theorem 6 *If $ABCD$ is a cyclic quadrilateral and if I_A, I_B, I_C, I_D are the incentres of the diagonal triangles BCD, CDA, DAB and ABC, respectively, then the points I_A, I_B, I_C and I_D form a cyclic quadrilateral.*

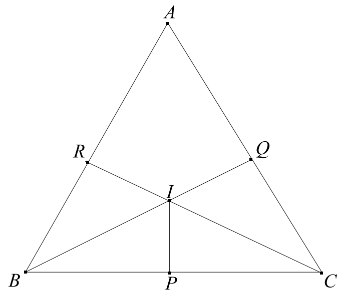


Figure 19:

Proof Recall that if ABC is a triangle, I is the incentre and P, Q, R are feet of perpendiculars from I to the sides BC, CA and AB then

$$
\begin{aligned}
B\widehat{I}C &= \frac{1}{2}(RIP + PIQ) \\
&= \frac{1}{2}(360^\circ - RIQ) \\
&= \frac{1}{2}(360^\circ - 180^\circ + R\widehat{A}Q) \\
&= 90^\circ + \frac{R\widehat{A}Q}{2}.
\end{aligned}
$$

Now apply this to the triangles BCD and ACD. We get (Figure 20) :

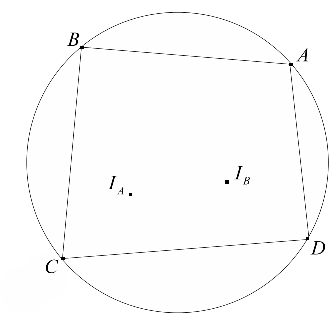


Figure 20:

$$
\begin{aligned}
C\widehat{I_A}D &= 90^\circ + \frac{1}{2}(C\widehat{B}D), \\
C\hat{I}_BD &= 90^\circ + \frac{1}{2}(C\widehat{A}D).
\end{aligned}
$$

But $C\widehat{B}D = C\widehat{A}D$ so it follows that $C\widehat{I_A}D = C\widehat{I_B}D$. Thus I_ACDI_B is a cyclic quadrilateral. The proof is now completed as in previous theorem.

Theorem 7 *If $ABCD$ is a cyclic quadrilateral and G_A, G_B, G_C and G_D are the centroids of the diagonal triangles BCD, CDA, DAB and ABC, respectively, then the quadrilateral $G_AG_BG_CG_D$ is similar to $ABCD$. Furthermore, the ratio of the lengths of their corresponding circles is $\frac{1}{3}$.*

Proof Recall that the quadrilateral formed by the orthocentres $H_A H_B H_C H_D$ is congruent to $ABCD$.

We also have the fact that all four diagonal triangles have a common circumcentre which is the centre of the circle $ABCD$. Let this be denoted by O.

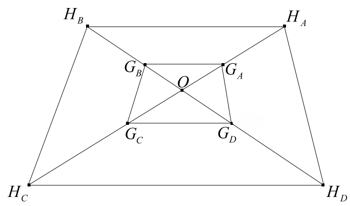


Figure 21:

Now join O to H_A, H_B, H_C and H_D. (Figure 21) The centroids G_A, G_B, G_C and G_D lie on these line segments and

$$\frac{|OG_A|}{|OH_A|} = \frac{|OG_B|}{|OH_B|} = \frac{|OG_C|}{|OH_C|} = \frac{|OG_D|}{|OH_D|} = \frac{1}{3}$$

Then it follows that $G_A G_B G_C G_D$ is similar to $H_A H_B H_C H_D$. The result now follows.

Finally we have

Theorem 8 *Let $ABCD$ be a cyclic quadrilateral and let A_1 and C_1 be the feet of the perpendiculars from A and C, respectively, to the diagonal BD and let B_1 and D_1 be the feet of the projections from B and D onto the diagonal AC. Then $A_1 B_1 C_1 D_1$ is cyclic.*

Proof Consider the quadrilateral BCB_1C_1 (Figure 22). Since

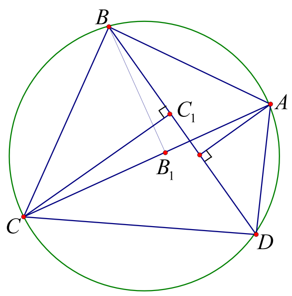


Figure 22:

$B\widehat{C_1}C = B\widehat{B_1}C = 90°$, then BCB_1C_1 is cyclic.

Similarly, A, A_1, B_1, B are cyclic, C, C_1, D_1, D are cyclic and A, A_1, D, D_1 are cyclic. Then by results of Proposition 1, $A_1B_1C_1D_1$ is cyclic.

Four Circles Theorem

If 4 circles are pairwise externally tangent, then the points of tangency form a cyclic quadrilateral.

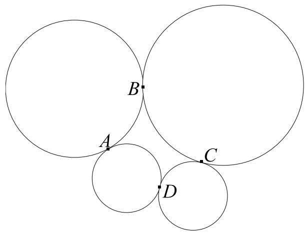


Figure 23:

In Figure 23, the quadrilateral $ABCD$ is cyclic.

<u>Remark</u> A similar theorem could not be true for 5 circles as 3 of the intersection points may lie on a line.

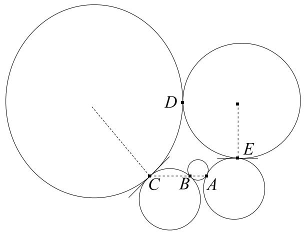
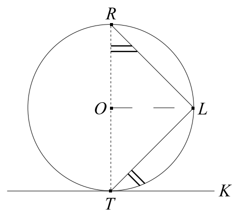


Figure 24: ABC lie along a line

<u>Proof</u> Recall that if TK is tangent to circle at T and O is centre of circle, then the angle between chord TL and tangent line T is one half of angle subtended at centre O by chord TL, i.e. $KTL = \frac{1}{2}(T\widehat{O}L)$. This is because $K\widehat{T}L = T\widehat{R}L$ where TR is the diameter at T.



Draw tangent lines AA_1, BB_1, CC_1 and DD_1 at points of contact with A_1, B_1, C_1, D_1 being points in region bounded by the circles. Thus

$$\begin{aligned} & B\widehat{A}D + B\widehat{C}D \\ &= B\widehat{A}A_1 + A_1\widehat{A}D + B\widehat{C}C_1 + C_1\widehat{C}D \\ &= \frac{1}{2}(A\widehat{O}_1B + A\widehat{O}_2D + B\widehat{O}_3C + C\widehat{O}_4D) \\ &= \frac{1}{2}(\text{sum of angles of quadrilateral } O_1O_2O_3O_4) \\ &= 180°. \end{aligned}$$

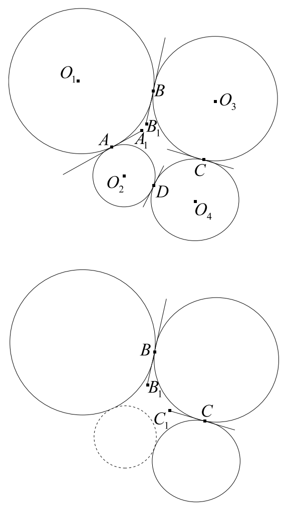


Figure 25:

Here we used the fact that the point of tangency of a pair of circles lies on the line joining their centres.

Remark If r_1, r_2, r_3 and r_4 denote the radii of the four circles then

$$\begin{aligned} &O_1O_2 + O_3O_4 = r_1 + r_2 + r_3 + r_4 \\ \text{and}\quad &O_1O_3 + O_3O_4 = r_1 + r_2 + r_3 + r_4 \end{aligned}$$

Thus $O_1O_2O_3O_4$ is a quadrilateral with the sums of the opposite side lengths equal. Such a quadrilateral is called *inscribable*, i.e. it has an incircle. In this situation, the circumcircle of $ABCD$ is the incircle of $O_1O_2O_3O_4$.