TILINGS

1.

Set-up: Cut out regular polygons of equal size length and various numbers of sides (1-10; about 12 of each type).

Definition 1. A polygon is called regular if all of its sides are equal and all of its angles are equal.

You can make some with Geogebra, the regular polygon button which is accessed by clicking in the lower right corner of the triangle (polygon) button.
http://www.geogebra.org/cms/
Let the circle participants try to build tilings: first with just on type of regular polygon, then combining two types, (some will probably try three types too) and let them record which combinations work. Some questions will arise naturally, like: why can't one use pentagons to build a tiling? why is it that some combinations never work?

Definition 2. (Wikipedia) A tessellation or tiling of the plane is a collection of plane figures that fills the plane with no overlaps and no gaps.

The desired results should look like some of these
http://mathworld.wolfram.com/Tessellation.html
The classic two-dimensional picture of a beehive is a tiling made out of regular hexagons. This makes the beehive into a sturdy construction, comfortable for the bees and suitable for their communal life. But why don't the bees construct their beehives out of pentagonal, or octagonal shapes? In mathematical terms, we could pose this question as follows:

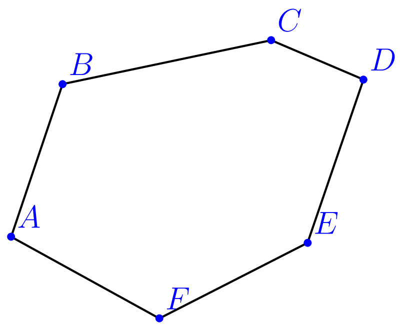
Question 1: For which integer numbers $n \geq 3$ does there exist a tiling of the plane by identical regular n-sided polygons?

Idea for solution: To approach this problem, we assume that there exists a tiling by identical regular n-sided polygons, and look at all the angles around a vertex A of the tiling. Their sum is $360°$, and they are all equal to each other. It would be helpful if we knew the size of one such angle:

Question 2: What is the size of an interior angle of a regular polygon with n sides?

For this, it would be helpful if we knew

Question 3: What is the sum of all the sizes of all interior angles of a (convex) polygon with n sides?



Finally something we can start to answer right away:

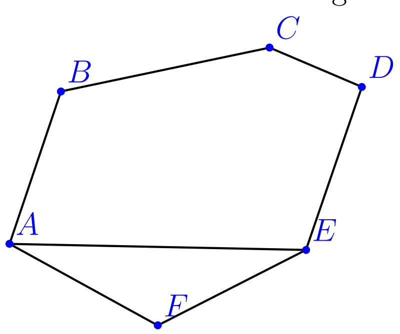
Answer 3: Let $S(n)$ = the sum of all the sizes of all interior angles of a polygon with n sides.

n	$S(n)$
3	$180°$
4	$2 \times 180° = 360°$
5	$3 \times 180° = 540°$

Indeed, $S(3)$ is widely known; the interior of a quadrilateral can be split into 2 triangles, and a pentagon into 3 triangles. We can thus prove

$$S(n) = (n-2)180° \tag{1.1}$$

by induction on n. This particular induction can be simple enough and appealing for kids. For the induction step, we assume $S(n) = (n-2)180°$ and need to prove $S(n+1) = (n-1)180°$. For this, we split the interior of a polygon of $(n+1)$ sides into one of n sides and a triangle.



Thus $S(n+1) = S(n) + 180°$ from our construction, and $S(n) = (n-2)180°$ from our assumption. Putting these together we get $S(n+1) = (n-1)180°$. Now we have

Question 2: What is the size of an interior angle of a regular polygon with n sides?

Answer 2: An interior angle of a regular polygon with n sides measures $\frac{(n-2)}{n}180°$:

n	angle of a regular polygon with$n$ sides
3	$60°$
4	$90°$
5	$108°$
6	$120°$.

Question 1: For which integer numbers $n \geq 3$ does there exist a tiling of the plane by identical regular n-sided polygons?

Answer 1: We return now to our tiling by identical regular n-sided polygons, and assume that there are k angles around a vertex A of the tiling, for an unknown integer k. Answer 2: above tells us that $A = \frac{(n-2)}{n}180°$ and so looking at the angles around any one vertex of the tiling leads to the equation

$$k\frac{(n-2)}{n}180° = 360°, \text{or, after simplifying, } k = \frac{2n}{n-2}.$$

At first sight this doesn't look too encouraging, since we know neither k nor n. However, we know two important facts:

(1) that both k and n are integers, and
(2) that $n \geq 3$ and $k \geq 3$.

Each of the facts above yields a method of solving the equation.

Due to (2), we can check the possible values of k and n case by case, starting with $n = 3$ and ending with $n = 6$. After that, $n > 6$ will imply $k < 3$ (For younger students, just concede that this is intuitively clear. For more advanced students: prove this, for example by showing that the function $f(x) = \frac{2x}{x-2}$ is decreasing for $x > 2$!). You will find that k is an integer exactly in the cases $n = 3$, 4 or 6.

Alternatively, due to (1), we can employ divisibility arguments. Indeed, since $k = \frac{2n}{n-2}$ is a whole number, then the fraction $\frac{2n}{n-2}$ can be simplified by $n-2$. But let's look at any common factor d for this fraction. Then

$$\left.\begin{array}{rcc} & & d|2n \\ \text{and } d|(n-2) & \Rightarrow & d|2(n-2) \end{array}\right\} \Rightarrow d|\,(2n-2(n-2)) \Rightarrow d|4.$$

In particular, since we know $(n-2)$ is a positive factor for the fraction, we get $(n-2) \in \{1, 2, 4\}$. This yields three cases as below

n	k
3	6
4	4
6	3

Question 4: Find all pairs of integer numbers $m > n \geq 3$ such that there exists a tiling of the plane made only of regular polygons of n sides and those of m sides. For each pair, make a sketch of a possible tiling.

Hint: Solve the equation $k\frac{n-2}{n} + l\frac{m-2}{m} = 2$ case by case. Note that in this case $k, l \geq 1$. Start with the case $k = 1$, $n = 3$. Then $l\frac{m-2}{m} = \frac{5}{3}$ or, equivalently, $\frac{m-2}{m} = \frac{5}{3l}$. When $l = 1, 3$ or 4, the equation has no integer solution m. When $l = 2$, we get $m = 12$. When $l = 5$ we get $m = n = 3$. If $l > 5$ then $m < 3$ which contradicts our basic assumption about m.

Continue with the case $k = 2$, $n = 3$ etc. Here are all possible solutions:

k	n	l	m
1	3	2	12
2	3	2	6
3	3	2	4
4	3	1	6
1	4	2	8.

And nice pictures can be found here:

http://mathworld.wolfram.com/Tessellation.html

2.

Question 5: (For more advanced students): Construct regular polygons of $n = 3$, 4, 5, 6, 8, 10, 12 sides using a straight edge and a compass.

Idea: The most interesting case is $n = 5$. We need to divide a circle in 5 equal arcs. In this case one such arc measures $2\pi/5$. One can calculate $x = \cos\frac{2\pi}{5}$ and $y = \cos\frac{\pi}{5} = -\cos\frac{4\pi}{5}$ by a nice trick, using the double angle formula for cosine. Thus

$$x = 2y^2 - 1 \text{ and } -y = 2x^2 - 1.$$

Subtracting the two equations: $x + y = 2(y^2 - x^2) = 2(y - x)(x + y)$ and since $x + y \neq 0$, we simplify; $1/2 = y - x$. Substitute $y = 1/2 + x$ above to get a quadratic equation in x.

We'll need to construct $\sqrt{5}$ which can be done as hypothenuse in a right-angles triangle with sides 2 and 1.