Numbers and squares

Set-up: teams of 3-4. Some cut-out squares and rectangles will be needed as seen below. A 3D cube model as sketched below also most useful.

Introduction: (Works best on 13/01/2012, black cat included). The number 13 has some very strange properties.

For example: $13^2 = 169$.
Now read 13 backwards, and square: $31^2 = 961$.

1. **Can you find other two-digit numbers with the same property?** (team contest, who can find more).
(Hint: try small digits).

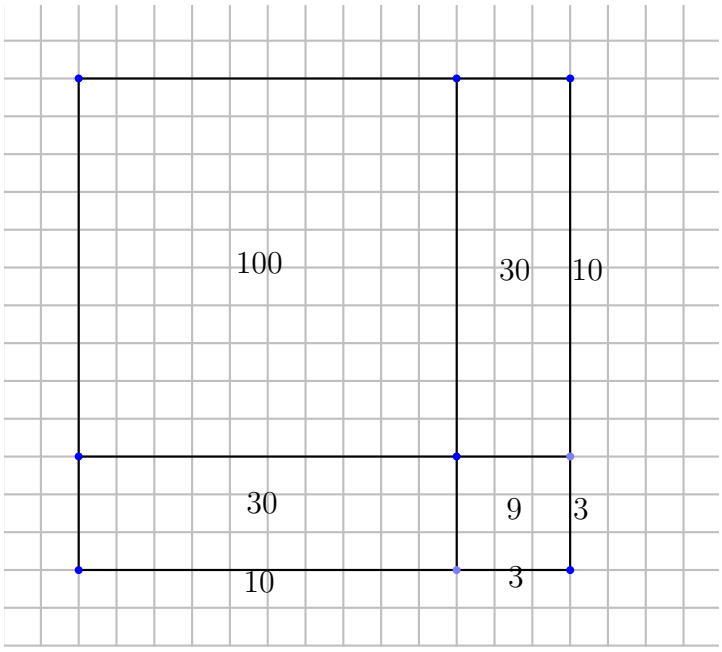
Answer:

$$11^2 = 121, \quad 11^2 = 121;$$
$$12^2 = 144, \quad 21^2 = 441;$$
$$22^2 = 484, \quad 22^2 = 484;$$
$$10^2 = 100, \quad 01^2 = 001.$$

(Kids can decide whether the last one is acceptable or not).

2. Try as they might, the teams **will not be able to find other numbers with a similar property**. They might be curious **why**. They may come up with some intuitively correct reasons, at least for why some of the other numbers don't work.

Suggest thinking about products as areas, so of square numbers as, well, areas of squares:
13^2 = the area of a square with side of length 13 = how many little squares of side length 1 form a square of side length 13.
Now splitting the numbers into tens and digits:
$13^2 = (10 + 3)^2 = 100 + 2 \cdot 30 + 9 = 169$.



Similarly, $31^2 = (30 + 1)^2 = 900 + 2 \cdot 30 + 1 = 961$.
Because 3 and 1 are small enough so that $3^2 = 9$ and $1^2 = 1$, and $2 \cdot 3 \cdot 1 = 6$ are all

digits: In the first case, 9 alone contributes to the unit digit, 6 alone to the tens digit, and 1 alone to the hundreds digit, whereas for the second number 1 alone contributes to the unit digit, 6 alone to the tens digit, and 9 to the hundreds digit. This happy situation no longer holds if either the square of the unit digit, or the square of the tens digit, or twice the tens times the unit digits are no longer digits...

3. Team challenge, or homework: **Try to find all three digit numbers such that if we read them backwards and then square we get their square, read backwards.** No calculators, please. Rather, use the squaring trick above.

However if patience is running thin replace no.3 with simple speed calculations: $301^2; 104^2$ etc.

4. **Abstract wrap-up:** $(a+b)^2 = a^2 + 2ab + b^2$. **Demonstrate this with two cut-out squares and two rectangles.** (It's important for clarity that a and b are different enough from each other).
(Of course this could also be proven formally by distributivity).

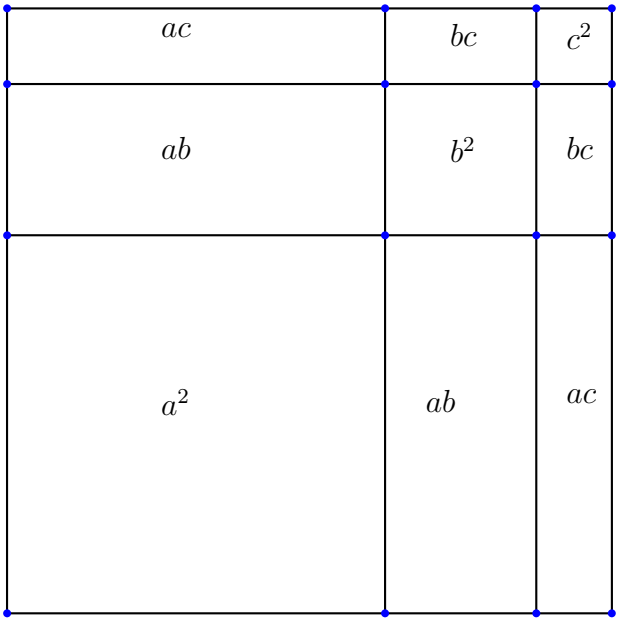
Can the teams use the cut-out figures above to come up with a formula for $(a-b)^2$?

Idea: Suppose $a > b$. When trying to fit the two rectangles of area ab into the square of area a^2, they will leave an empty square of area $(a-b)^2$, but on the other hand they have to overlap over an area of b^2. It remains to interpret this overlap correctly into a formula.

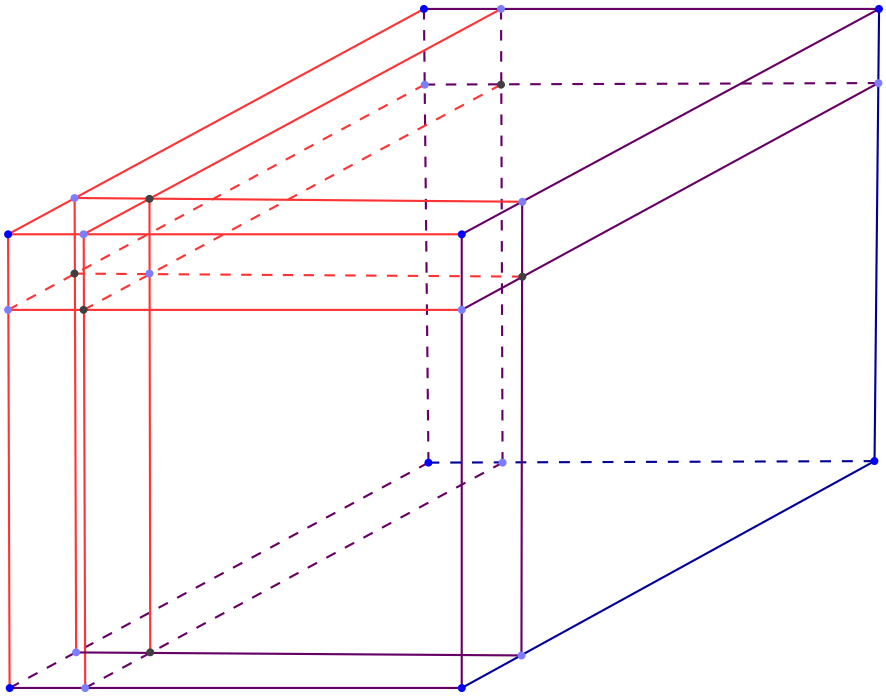
5. You can impress your friends with **speed calculations based on the formula** $(a-b)^2 = a^2 + b^2 - 2ab$. Team contest: Allow for use of pen and paper in the first few instances then move on to mental arithmetic.
$19^2 \quad 49^2 \quad 99^2 \quad 29^2 \quad 199^2 \quad 299^2$...

6. More abstract wrap-up: **find a formula for** $(a+b+c)^2$ **using diagram or cut-out squares and rectangles of side lengths** a**,** b**,** c. How about $(a+b+c+d)^2$?

Answer: $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$.



7. Formula for $(a+b)^3$ using cubes and parallelepipeds. More difficult: $(a-b)^3$.
Answer:$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.



We have a 3D model or building material for your use.
$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$.
8. **Quick calculations:** 11^3, 101^3, 201^3, 99^3 (for the brave ones).
9. **How many cubes can you find which read well backwards like before?**