Constructions with a broken ruler and some coins.

You have a ruler with two parallel sides, but broken so that it has no right angles. You also have many coins. You draw some circles of various sizes with your coins. Some are so small that they can fit completely under your ruler, others are larger.

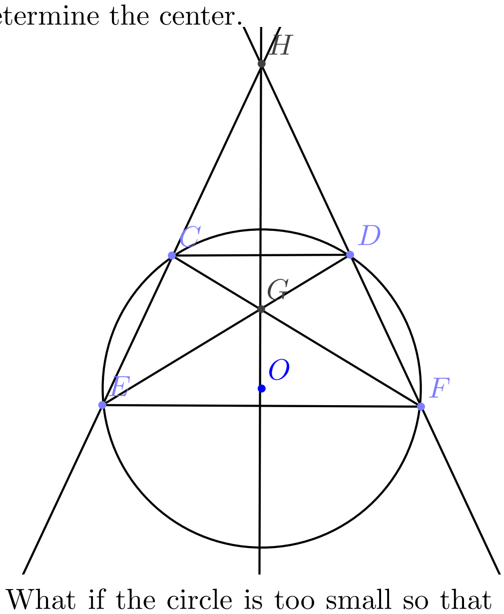
Questions:

(1) Find the center of each circle.
(2) Draw some random segments. Can you find their midpoint?
(3) Can you extend your segments to double their initial length?
(4) Can you split your segments into 3 (4, 5, etc.) equal pieces?
(5) Can you drop a perpendicular from a given point to a given line?
(6) Can you draw a square given one of its sides?
(7) Are there things that you cannot do?

Hints:

(1) We want to draw some diameters for your circles, so, some figures on the circle which have axes of symmetry.

If our circle is large enough that it doesn't fit under the ruler, we're lucky, because we can draw two parallel lines intersecting the circle. They cut the circle under an isosceles trapezoid, which has an axis of symmetry. Then we can draw another isosceles trapezoid in the circle, so another axis of symmetry. That's enough to determine the center.

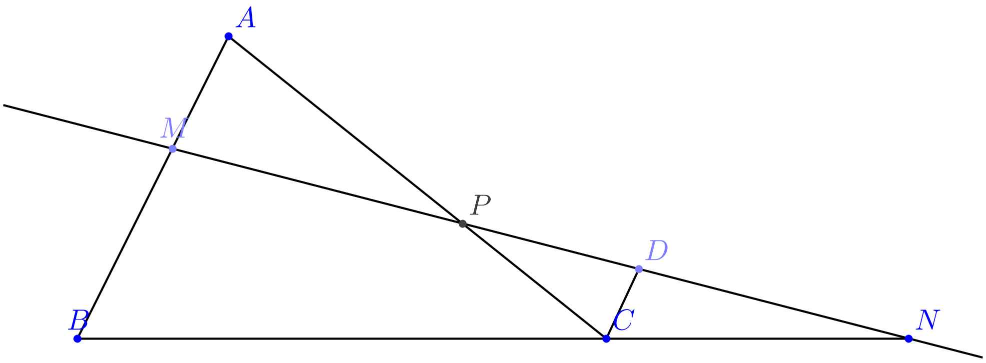


What if the circle is too small so that no isosceles trapezoid can be drawn in it with the given ruler? We can draw two parallel lines, one of which intersects the circle. If through any given point we could draw a line parallel to the two, we would win!

We'll need to apply **Menelaus' theorem**, which tells us when three points on the sides of a triangle are collinear.

Theorem 0.1. *M, N, P are collinear if and only if*

$$\frac{|AM|}{|MB|}\frac{|BN|}{|NC|}\frac{|CP|}{|PA|} = 1$$



Proof. If the points are collinear, then to prove relation: Let $CD \parallel AB$. Then multiply $\frac{|BN|}{|NC|} = \frac{|MB|}{|CD|}$ with $\frac{|CP|}{|PA|} = \frac{|CD|}{|AM|}$.

If we know relation, assume points are not collinear, let M' be the intersection of lines NP and AB. It has to lie inside the segment $[AB]$ just like M. Then by the previous argument,

$$\frac{|AM'|}{|M'B|}\frac{|BN|}{|NC|}\frac{|CP|}{|PA|} = 1$$

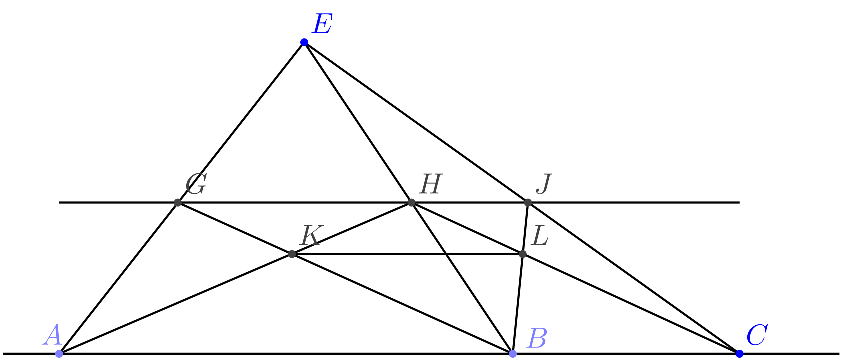
while by assumption

$$\frac{|AM|}{|MB|}\frac{|BN|}{|NC|}\frac{|CP|}{|PA|} = 1$$

hence $\frac{|AM'|}{|M'B|} = \frac{|AM|}{|MB|}$ so $\frac{|AM'|}{|AB|} = \frac{|AM'|}{|AM'|+|M'B|} = \frac{|AM|}{|AM|+|MB|} = \frac{|AM'|}{|AB|}$ so $|AM| = |AM'|$ thus M is M'.

□

Here's how to draw a line through K parallel to AB and GH:



The proof that $KL \parallel AB$ follows from Thales' theorem together with Menelaus' theorem applied to collinear points A, K, H on the one hand side, and C, L, H on the other hand.

(2) In the above picture, the line EK intersects GH and AB at their midpoints. (application to Menelaus' theorem or alternatively Ceva's theorem.)
(3) We know how to construct midpoints, and parallels. We can construct midlines as well. To extend $[AB]$, let $[AC]$ a segment on another line, M its midpoint. Draw a parallel to MB through C and intersect with the line AB. We can now extend a segment by any multiple we want, and applying Thales'theorem, we can also split any segment in any number of parts we need.
(4) Midpoints, symmetry axes, circles, parallels ... yes we can drop perpendiculars,
(5) ...thus also construct squares...
(6) I don't think we can bisect angles...

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