# Eclectic Equations

This lesson plan works well for 2nd year students who haven't learned simultaneous equations in school.  3rd year students should have no difficulty solving the questions up to questions 10-12 and can could be encouraged to work ahead until they get to those.

## Resources:

1. One question sheet per student.

## Aims:

The purpose of this lesson is to discuss the use of equations and systems of equations in the solution of word problems. The aim is to get students used with abstract notations and to a conceptual way of understanding division with remainder.

## Eclectic Equations

**Challenge 1 – Things Are Hotting Up!**
Kevin and Michelle are cousins. Kevin lives in the New York, while Michelle lives in Cork. Michelle wants to tell Kevin that it is 10 degrees Celsius in Cork, but in the United States temperature is given in degrees Fahrenheit. She learns that if you divide a temperature in Celsius by 5, multiply by 9, and add 32, you get a temperature in Fahrenheit.

(i) What's the temperature in Cork in Fahrenheit?

$10°C \xrightarrow{\div 5} \boxed{\phantom{00}} \xrightarrow{\times 9} \boxed{\phantom{00}} \xrightarrow{+32} \boxed{\phantom{00}\ °F}$

(i) Kevin now wants to tell Michelle that it's 68 degrees Fahrenheit. How many degrees Celsius is 68 Fahrenheit?

$\boxed{\phantom{00}\ °C} \leftarrow \boxed{\phantom{00}} \leftarrow \boxed{\phantom{00}} \leftarrow \boxed{68\ °F}$

(ii) Absolute zero is, roughly speaking, the theoretical lowest possible temperature. If it is –459.67 degrees Fahrenheit, calculate it in degrees Celsius.

**Challenge 2 - gnikroW**

The mother of triplets, Annie, Benny, and Jenny, left some strawberries for them on the table. First, Annie took a third. Later, Benny took a third of what was left. Finally, Jenny took 4 strawberries, which was a third of the remaining amount. How many strawberries did the mother leave for them?

**Challenge 3**
A length of pipe is broken into 5 parts – the second is twice as long as the first, the third is twice as long as the second, the fourth is twice as long as the third, and the fifth is twice as long as *all the other parts put together*. If the pipe is 360cm long, find the length of each part.

**Challenge 4**
There were 64 sweets in a tin of Roses, surrounded by four teens. A couple of minutes later, all but 5 of the sweets have been devoured. Beatrice has eaten 5 fewer sweets than Abby, Casey has eaten twice as many as Beatrice, and Dana has eaten as many as Abby and Casey together. How many sweets did each of them eat?

**Challenge 5**
Miah is twice as old as Alice today, but 6 years ago on the same day he was three times as old as Alice was then. How old are Miah and Alice today? (and if you can solve that without rereading the question, then fair play!)

**Challenge 1 revisited**

(iv) Is there a temperature that's the same in Celsius and Fahrenheit? If so, find it!

## Subtle Substitutions

**Challenge 6**
If 3 grapefruits and 2 pineapples weigh the same as 5 grapefruits and 1 pineapple, and 1 grapefruit and 2 pineapples weight 12 kgs, how much does a grapefruit weigh? How about a pineapple?

**Challenge 7**
In Barcaland, the denominations of currency are the Messi, the Xavi, and the Iniesta. There is a complicated relationship between them:

1 Messi and 4 Xavis are worth the same as 3 Iniestas and 5 Xavis

2 Iniestas and 7 Xavis are worth the same as 4 Iniestas and 4 Xavis

If 1 Messi is worth €99 million euro, find (in €) the value of each of the other denominations.

**Challenge 8**
One hot summer day, four friends bought 3 large vanilla ice-creams and 5 medium strawberry ice-creams for €15. The next day they bought 6 large vanilla ice-creams and 2 medium strawberry ice-creams for €18. How much does each type of ice-cream cost?

## Mind your Q's and R's!

**Challenge 9**
a) When we divide a number by 7 we get a remainder of 3. What remainder will we get if we double the number, then divide it by 7?

b) When we divide a second number by 7 we get a remainder of 4. What remainder will we get if we triple the number, then divide it by 7?

c) When we divide a third number by 7 we get a remainder of 6. What remainder will we get if we quadruple the number (i.e. multiply it by 4), then divide it by 14?

**Challenge 10**
The sum of three natural numbers is 85. When we divide the first number by the second number we get the quotient 2, leaving the third number as remainder. Find out the numbers knowing that the second number is 5 times as big as the third.

**Challenge 11**
In a room there are 7 paper boats, an unknown number of people and much too much paper. Each person makes 2 new paper boats. Then a new person comes in and distributes the boats equally among all the people in the room, leaving the remainder on the table. What are all possible numbers of boats on the table?

**Challenge 12**
The double of a mystery number M (larger than 3) is divided by the difference between the mystery number M and the number 3, leaving a remainder of 2. Find all possible values of M.

## Eclectic Equations

**Challenge 1 – Things Are Hotting Up!**

Kevin and Michelle are cousins. Kevin lives in the New York, while Michelle lives in Cork. Michelle wants to tell Kevin that it is 10 degrees Celsius in Cork, but in the United States temperature is given in degrees Fahrenheit. She learns that if you divide a temperature in Celsius by 5, multiply by 9, and add 32, you get a temperature in Fahrenheit.

(i) What's the temperature in Cork in Fahrenheit?

(i) Kevin now wants to tell Michelle that it's 68 degrees Fahrenheit. How many degrees Celsius is 68 Fahrenheit?

(ii) Absolute zero is, roughly speaking, the theoretical lowest possible temperature. If it is –459.67 degrees Fahrenheit, calculate it in degrees Celsius.

There are many ways of approaching each of these problems. Note that Trial and Improvement is included for Challenges 2 and 3, as this tends to be the most common approach favoured by younger students.

***Solution:***

(i) $[(10 \div 5) \times 9] + 32 = 50°\mathrm{F}$

(ii)
1. Subtract 32
2. Divide by 9
3. Multiply by 5

So $[(68 - 32) \div 9] \times 5 = 20°\mathrm{C}$

(iii)

**Challenge 2 – gnikroW**

The mother of triplets, Annie, Benny, and Jenny, left some strawberries for them on the table. First, Annie took a third. Later, Benny took a third of what was left. Finally, Jenny took 4 strawberries, which was a third of the remaining amount. How many strawberries did the mother leave for them?

*Work from the end backwards:*

When Jenny came along: 4 strawberries = 1/3 of amount
So there were 12 strawberries left.

When Benny came along: He took 1/3 of what was there, leaving 12 behind.
So he left 2/3, which was 12 strawberries.
So there must have been 18 when he came.

When Annie came along: She took 1/3 of what was there, leaving 18 behind.
So she left 2/3, which was 18 strawberries.
So there must have been 27 when she came.

*Working forwards with x:*

Start with $x$ strawberries.

When Annie leaves, there are 1/3 gone, so there are $\frac{2}{3}(x)$ remaining.

When Benny leaves, there are 1/3 of these gone, so there are 2/3 remaining.

$\frac{2}{3}$ of $\frac{2}{3}(x)$ is $\frac{2}{3}\left(\frac{2}{3}(x)\right) = \frac{4}{9}(x)$ .

Annie takes 1/3 of these, which is 4. So $\frac{1}{3}\left(\frac{4}{9}(x)\right) = 4$, i.e. $x = 27$.

**Challenge 3**

A length of pipe is broken into 5 parts – the second is twice as long as the first, the third is twice as long as the second, the fourth is twice as long as the third, and the fifth is twice as long as *all the other parts put together*. If the pipe is 360cm long, find the length of each part.

*Visual Solution:*

As the first length is the smallest, we can represent the pipe as follows:

|-|--|----|--------|------------------------------|

Altogether there are 45 pieces here. Thus $45$ pieces $= 360$ cm, so each piece is 8cm.

*Algebraic Solution:*

Let $x$ stand for the length of each piece. Then $x + 2x + 4x + 8x + 30x = 360$.

**Challenge 4**

There were 64 sweets in a tin of Roses, surrounded by four teens. A couple of minutes later, all but 5 of the sweets have been devoured. Beatrice has eaten 5 fewer sweets than Abby, Casey has eaten twice as many as Beatrice, and Dana has eaten as many as Abby and Casey together. How many sweets did each of them eat?

*Trial and Improvement:*

Take a guess – say, Abby eats 10 sweets. Then Beatrice has eaten 5, Casey has eaten 10, and Dana has eaten 20. These add to 45. As 59 sweets have been eaten, we are 14 short. If we give Abby an extra sweet, then Beatrice gets 1 extra, Casey gets 2 extra, and Abby gets 3 extra – that is, 7 extra sweets in total. So we need to give Abby 2 extra sweets, in order for the required amount to be eaten. So Abby eats 12 sweets, and we can easily find the rest.

*Visual Solution:*

We could notice that Beatrice ate the least amount. We can represent Beatrice's share as one circle (her plate), then Abby has a full plate + 5 dots for the candies etc.

***A*** ***B*** ***C*** ***D***

This gives 7 plates of sweets, and 10 extra sweets. As these add to 59, the 7 plates must add to 49. So each plate has 7 sweets on it.

This preempts the transition to equations, where we replace each plate with an $x$.

*Algebraic Solution:*

Beatrice has $x$ sweets, Abby has $x + 5$, Casey has $2x$ and Dana has $2x + x + 5$.

The equation is then $x + x + 5 + 2x + 2x + x + 5 = 59$, which ssolves to give $x = 7$.

**Challenge 5**

Miah is twice as old as Alice today, but 6 years ago on the same day he was three times as old as Alice was then. How old are Miah and Alice today?

*Trial and Improvement:*
We could guess that Miah is 20 and Alice is 10. In this case, 6 years ago they would have been aged 14 and 4, respectively. But $14 \div 4 = 3.5$, which is too big. We need to decrease the ratio between them, by making the initial ages bigger.
We'll try making their ages bigger – say, 24 and 12. Then 6 years ago their ages were 18 and 6 respectively, giving exactly the right ratio.

*Visual Solution:*
We could draw a timeline, where the length of the line represents the age in years.

| | Miah | Alice |
|---|---|---|
| Today | _______._______ | _______ |
| 6 years ago | ___._______ | ___ |

The difference between the two ages stays the same, but it is equal to Alice's age today and double Alice's age 6 years ago. So Alice's age doubled in 6 years, which means that it went from 6 to 12 now.

*Algebraic Solution:*
In equations, we could have, for instance, $x =$ Alice's age now. Then we have:

| | Today | 6 years ago |
|---|---|---|
| Miah | $2x$ | $2x - 6$ |
| Alice | $x$ | $x - 6$ |

The equation is $2x - 6 = 3(x - 6)$, as Miah's age 6 years ago is thrice Alice's, and solves to give $x = 12$. Check that the participants master the steps of solving the equation:
$2x - 6 = 3x - 18$. Add 18 on both sides and simplify:
$2x + 12 = 3x$. Subtract 2x on both sides:
$12 = x$.

**Challenge 1 Revisited**

(iv) Is there a temperature that's the same in Celsius and Fahrenheit? If so, find it!

Take a temperature, say $x$, in degrees Celsius. Then, in Fahrenheit, this is $\left[\left(\frac{x}{5}\right) \times 9\right] + 32$. As we want this answer to be equal to $x$, we have $\left[\left(\frac{x}{5}\right) \times 9\right] + 32 = x$, or equivalently

$$\frac{9}{5}x + 32 = x.$$

which solves to give $x = 40$. (Again, make sure that the participants master all steps of the solution).
Thus $40°\mathrm{F} = 40°\mathrm{C}$.

## Subtle Substitutions

**Challenge 6**

If 3 grapefruits and 2 pineapples weigh the same as 5 grapefruits and 1 pineapple, and 1 grapefruit and 2 pineapples weight 12 kgs, how much does a grapefruit weigh? How about a pineapple?

*Solution:*
Imagine the fruits on the two scales of the balance. So:

$$G + G + G + P + P = G + G + G + G + G + P$$

To simplify this, we could remove one $P$ and three $G$s from each side, to get:

$$P = G + G$$

The second equation gives:

$$G + P + P = 12$$

Now, we can replace each $P$ in the second equation with two $G$s, (just as Indiana Jones does as the start of Raiders of the Lost Ark), to get:

$$G + (G + G) + (G + G) = 12$$

Thus $G = 2.4$ kg, and $P = 4.8$ kg.

**Challenge 7**

In Barcaland, the denominations of currency are the Messi, the Xavi, and the Iniesta. There is a complicated relationship between them:

1 Messi and 4 Xavis are worth the same as 3 Iniestas and 5 Xavis

2 Iniestas and 7 Xavis are worth the same as 4 Iniestas and 4 Xavis

If 1 Messi is worth €99 million euro, find (in €) the value of each of the other denominations.

*Solution:*

We have

$$M + X + X + X + X = I + I + I + X + X + X + X + X$$

Removing what's common from both sides gives

$$M = 3I + X \qquad \text{...(1)}$$

Next

$$I + I + X + X + X + X + X + X + X = I + I + I + I + X + X + X + X$$

Which simplifies to

$$3X = 2I$$

This can be rearranged to give $\quad X = \frac{2}{3}I \qquad$ ...(2)

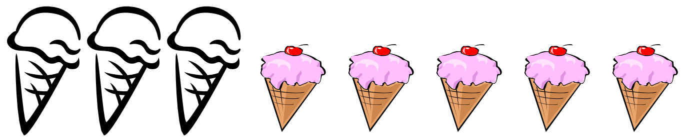
Substituting (2) into (1) gives $\quad M = 3I + \frac{2}{3}I = \frac{11}{3}I$

Finally, as $M = 99{,}000{,}000$, we have $\frac{11}{3}I = 99{,}000{,}000$, i.e. $I = €27{,}000{,}000$ and $X = €18{,}000{,}000$.

**Challenge 8**

One hot summer day, four friends bought 3 large vanilla ice-creams and 5 medium strawberry ice-creams for €15. The next day they bought 6 large vanilla ice-creams and 2 medium strawberry ice-creams for €18. How much does each type of ice-cream cost?

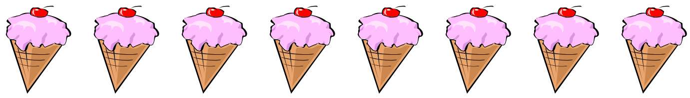
*Solution 1:*

 make €15.

 make €18.

The difference of €3 is the price difference between 3  and 3 , so  =  +1.

Substituting in the first equation:

 +3=15 so after simplifying

 $= \frac{12}{8}$=1.5 and  =2.5.

*Solution 2:*
Solve the simultaneous equations $3x + 5y = 15$ and $6x + 2y = 18$ by substitution.
Make sure the participants master all the steps.

## Mind Your Q's and R's!

**Challenge 9**
a) When we divide a number by 7 we get a remainder of 3. What remainder will we get if we double the number, then divide it by 7?
b) When we divide a second number by 7 we get a remainder of 4. What remainder will we get if we triple the number, then divide it by 7?
c) When we divide a third number by 7 we get a remainder of 6. What remainder will we get if we quadruple the number (i.e. multiply it by 4), then divide it by 14?

*First Solution:*

(i) Think of the number as:

$$7 + 7 + 7 + \cdots + 7 + 3$$

When we double it we get:

$$2(7) + 2(7) + 2(7) + \cdots + 2(7) + 2(3)$$

7 still divides into all the terms but the last. So we will be left with a remainder of 6.

(ii) Think of the number as:

$$7 + 7 + 7 + \cdots + 7 + 4$$

When we triple it we get:

$$3(7) + 3(7) + 3(7) + \cdots + 3(7) + 3(4)$$

7 still divides into all the terms but the last. So the only term contributing to the remainder is $3(4) = 12$. So the remainder will be 5.

(iii) Think of the number as:

$$7 + 7 + 7 + \cdots + 7 + 6$$

When we quadruple it we get:

$$4(7) + 4(7) + 4(7) + \cdots + 4(7) + 4(6)$$

14 divides into all the terms but the last. So the only term contributing to the remainder is $4(6) = 24$. So the remainder will be 10, when we divide by 14.

*Second Solution:*

(i) Think of the number as $7n + 3$, for some number $n$.
When we double it we get $14n + 6$.
7 divides into $14n$, so we will be left with a remainder of 6.

(i) Think of the number as $7n + 4$, for some number $n$.
When we triple it we get $21n + 12$.
7 divides into $21n$, so we will be left with a remainder of 5 when we divide 7 into 12.

(i) Think of the number as $7n + 6$, for some number $n$.
When we quadruple it we get $28n + 24$.
14 divides into $28n$, so we are left with a remainder of 10 when we divide 14 into 24.

**Challenge 10**

The sum of three natural numbers is 85. When we divide the first number by the second number we get the quotient 2, leaving the third number as remainder. Find out the numbers knowing that the second number is 5 times as big as the third.

*Visual Solution:*
We can represent the smallest number by one line segment, and the others by more:

| | | |
|---|---|---|
| Third Number | __ | 1 unit |
| Second Number | __.__.__.__.__ | 5 units |
| First Number | __.__.__.__.__.__.__.__.__.__.__ | 5 units + 5 units + 1 unit |

Altogether we have 17 units here, which add to 85. So each unit is worth 5.

*Algebraic Solution 1:*
Let $x$ be the third number, so that the second number is $5x$ and the first number is $2(5x) + x = 11x$. So $x + 5x + 11x = 85$.

*Algebraic Solution 2:*
Let the first, second and third numbers be $a$, $b$ and $c$ respectively.
Then $a = 2b + c$
But $a + b + c = 85$, and $b = 5c$.

A simple substitution now gives $a = 11c$ and $a + 6c = 90$, and another substitution gives $17c = 85$.

**Challenge 11**

In a room there are 7 paper boats, an unknown number of people and much too much paper. Each person makes 2 new paper boats. Then a new person comes in and distributes the boats equally among all the people in the room, leaving the remainder on the table. What are all possible numbers of boats on the table?

*Algebraic Solution:* Let's say there were $n$ people in the room initially. Then we divide $2n + 7$ boats to $n + 1$ people. Obviously $2n = 2 \times n$, so we can say

$$2n + 7 = 2(n + 1) + 5.$$

Hence we can hurry to say that the remainder is 5, but this is only true when

$$n + 1 > 5,$$

Since the remainder should always be smaller than the divisor. (We're only dealing with positive divisors so far with this audience).

So $n > 4$. We solve the other cases $n = 0, 1, 2, 3, 4$ separately and also get possible remainders of 0, 1, 2.

**Challenge 12**

The double of a mystery number M (larger than 3) is divided by the difference between the mystery number M and the number 3, leaving a remainder of 2. Find all possible values of M.

*Algebraic Solution:*

We first do the division like in the previous problem:

$2M = 2(M - 3) + 6.$

As the remainder is actually 2, not 6, that means that $M - 3$ fits into 4 an integer number of times. Also, if $M > 3$ we know $M - 3 > 2$ because the divisor should be larger than the remainder. So the only option is $M - 3 = 4$, hence $M = 7$.

We can't have $M = 3$.