Parity Party Returns,   Starting mod 2 games

Resources

- A few sets of dominoes – only for the break time!
- A few chessboards pieces: 2 for each student, to play the role of knights.
- Small coins, 16 per group of players.
- Activity sheets for students
- Solution sheets for tutors
- Paper hats (black and white) made of stapled A4 paper (optional).

Only hand in the chessboard and the coins at the time of solving the game and take them away immediately afterwards. We'll keep the dominoes for break time.

Parity Party II

1. Blackboard game

The numbers 1 to 2013 are written on a blackboard.

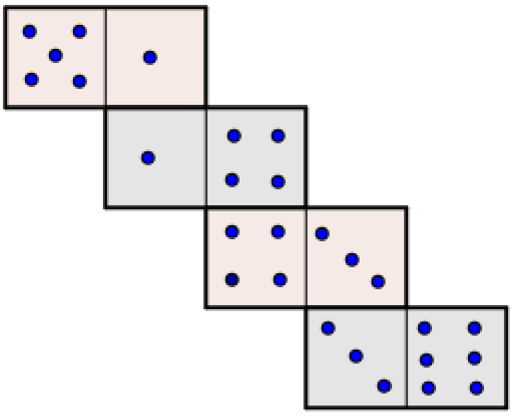
a) We can pick any two numbers from the blackboard, erase them and replace them with their sum. If we do this long enough, only one number will be left on the blackboard. What will that number be?

b) We start the game again, but this time we can pick any two numbers from the blackboard, erase them and replace them with their positive difference. If we do this long enough, is it possible for the only number on the blackboard to be zero?

2. More dominoes

A domino is a rectangle made up of two squares. Each square has 0 to 6 dots on it. A set contains one of each kind of domino.

a. How many dominoes are there in a set?
b. All the dominoes in a set are lined up in a chain (so that the number of dots on the ends of side-by-side dominoes match). If the square at one end has 5 dots on it, how many dots does the square on the other end have?
c. If we take out all the dominoes that have a square with no dot from a set, can we make a chain with the left over dominoes?



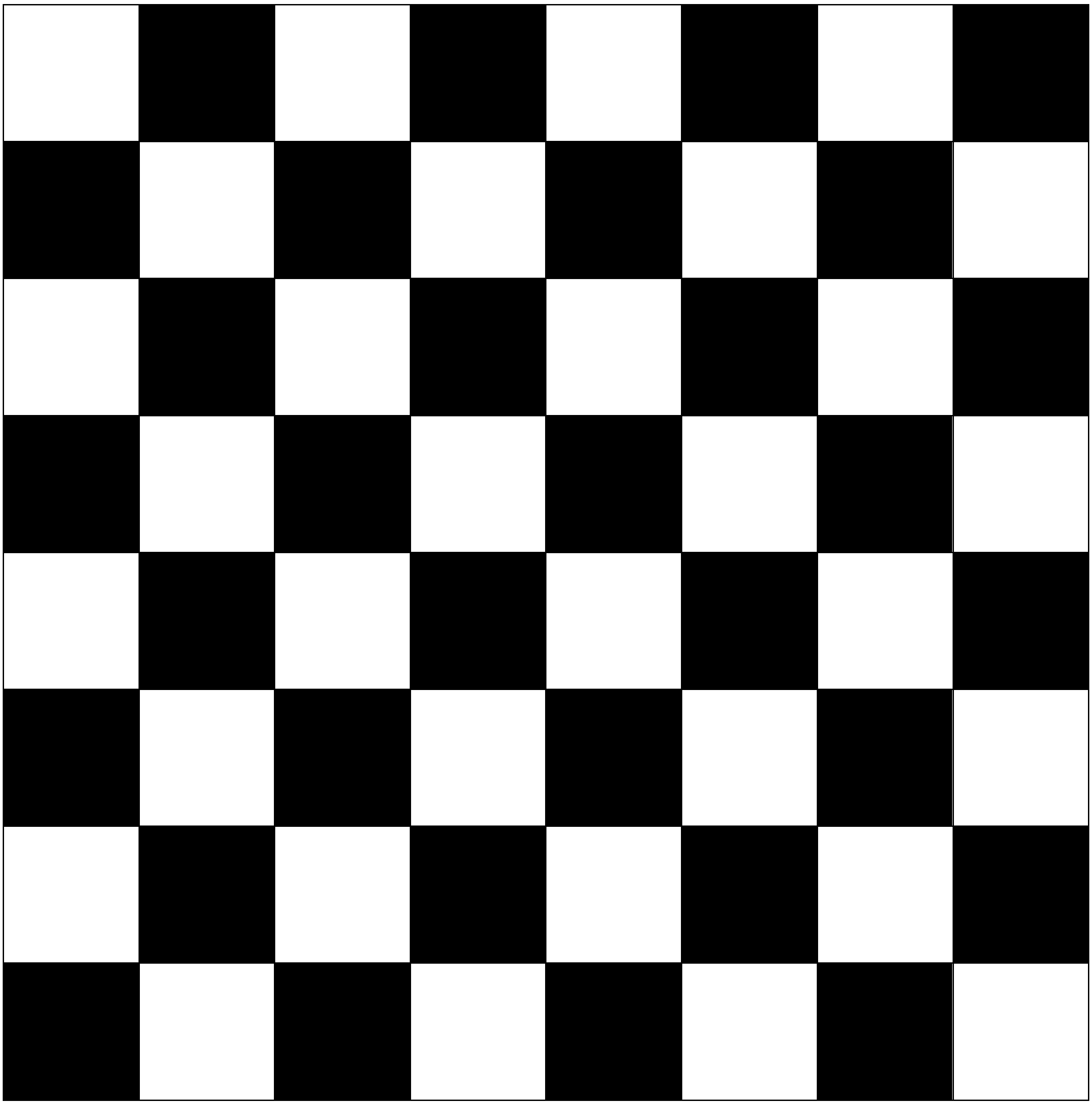
3. Chessboard problems

a) On a chessboard, a knight makes an “L-shaped” move, made up of moving 2 squares in 1 direction, and 1 square in another direction (not counting the starting square). If a knight starts on one corner square of the board, what is the fewest number of moves it takes for the knight to get to the opposite corner of the board?

b) Is it possible for the knight to get from one white corner to the opposite one while landing on every other square on the board exactly once?

c) Is it possible for the knight to be on any square on the board (after starting from the corner) after exactly 6 moves?

d) Is there any number, such that starting from a corner square, the knight can get to any square on the board it likes in exactly that many moves?



Modulo 2 Arithmetic

Remember the even/odd rules:

even+even =even
even+odd=odd
odd+odd= even

Sometimes people are too lazy to write “odd” and “even” and so write “1” for all odd numbers and “0” for all even numbers. This is called mod 2 arithmetic, and in one respect it is different from the usual sum rules:

Mod 2 arithmetic:

0+0=0
1+0=1
1+1=0

1+1=0 is like turning a light switch: if you turn it once, light is on.  If you turn it again, light is off.

1. Light switch game

Four Christmas light bulbs are arranged in a 2×2 grid. The switches for this grid are so connected that whenever you turn the switch for one of the light bulbs, this also affects the light bulbs immediately up or down, right or left from it. I’d like to know all the possible patterns of light I can get.

a) Examples:

Unlit grid:

0	0
0	0

In the unlit grid, turn the left upper corner light switch:

1	1
1	0

In the unlit grid, turn the right upper corner light switch:

1	1
0	1

In the unlit grid, turn the left lower corner light switch:

In the unlit grid, turn the right lower corner light switch:

Now let’s turn the switches in succession:

Left Up + Right Up:

1	1
1	0

+

1	1
0	1

=

Left Up + Right Up + Right Low:

+

+

=

b) If you start from the unlit grid, how many patterns can you get by turning switches (either one switch, or a few switches in succession)?

c) If you start from

1	0
0	0

how can you get to

0	0
1	0

?

d) If you start from

1	0
0	0

how can you get to

0	0
0	1

?

2. Coin game

One face shows the denomination (1 cent, 2 cent, 5 cent, 10 cent). Let's call this face "Cent".

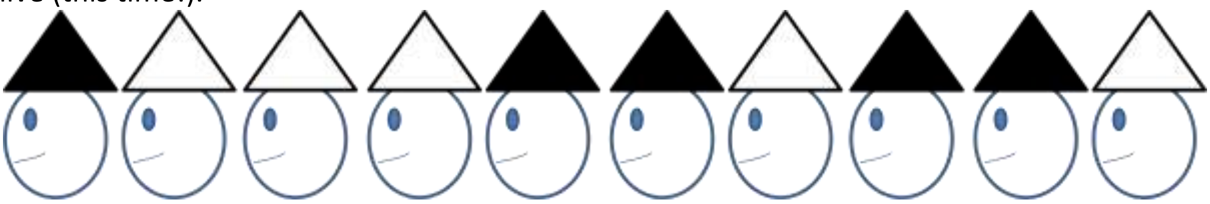
The other face has a 12 star circular border. Let's call this face "Star".

Arrange 16 coins on a table, so that they all have the same face up. One player is turned away from the table, eyes closed. Another player flips any of the coins on the board any number of times, each time saying "Tap", and at the end covers one of the coins with the hand. The first player turns around and has to guess which face of the covered coin is up.

Is there a strategy for guessing all the time?

3. Hat game

Being an extremely cruel and vicious maths teacher, I have decided to take ten of the students and stand them in a line one behind the other. On each of their heads I will place a hat, either black or white, I don't care how many of each colour. Each person will see the colours of all the hats ahead of them, but be unable to see those behind them or their own. The kids starting from the back must tell what colour hat they are wearing, if they pick the wrong colour unfortunately they will die. However if they pick the right colour I will let them live (this time!).



Before we start this whole process, the students are allowed to discuss and to come up with the best strategy in order to save as many as possible…and just so you know you can definitely save at least nine of the kids!!

3. Chocolate unwrapping game

http://funschool.kaboose.com/arcade/games/game_chocolate_biz.html

a) Can you describe the game in 0-s and 1-s? How would you represent each 1 move?

b) If you start with a completely wrapped chocolate and then click on each of its squares exactly once, what pattern do you get at the end?

c) Can you win the game in 4 moves?

Hint: Can you divide the chocolate in four non-overlapping areas, each turned on by 1 switch?

Parity Party II Solutions and Hints

1. Blackboard game

The numbers 1 to 2013 are written on a blackboard.

a) We can pick any two numbers from the blackboard, erase them and replace them with their sum. If we do this long enough, only one number will be left on the blackboard. What will that number be?

b) We start the game again, but this time we can pick any two numbers from the blackboard, erase them and replace them with their positive difference. If we do this long enough, is it possible for the only number on the blackboard to be zero?

Solution:

Hint: In both questions a) and b), the kids may feel more comfortable after playing the game with the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9.

a) This is a matter of "putting brackets around" two numbers in the sum. We know addition is associative and commutative, so no matter what, the answer is

$1+2+3+\cdots+2013=\frac{2014\times 2013}{2}=2{,}027{,}091$.

b) Hint: In the end, we are going to use all the numbers, some will come with + and others with $-$. Will the answer be even or odd?

Whether the answer is even or odd depends on how may odd numbers we have in the list

$1,\ 3,\ 5,\ldots,2013.$ After subtracting 1,

That's the same as counting $0,\ 2,\ 4,\ldots,2012$ After dividing by 2,

That's the same as counting $0,\ 1,\ 2,\ldots,1006.$

There are 1007 numbers in this list. (Note: there are 10 numbers in the list 0, 1, 2, ..., 9).

Answer: No. From 1 to 2013 (included) there are an odd number of odd numbers, so the answer will always be odd.

Extension question: What other numbers can we replace 2013 by to get the same answer?

Answer: Any multiple of 4 plus 1. Indeed, write such a number as 4k+1. There will be 2k+1 odd numbers in the list $1,\ 3,\ 5,\ \ldots,4k+1.$

Extension question: What if we replace 2013 by a number of the type N=4k+3? Try the cases N=3 and N=7 first.

Answer: If n=3 we have (3-2)-(1-0)=0. If n=7 we have (7-6)-(5-4)=0 as well, which combined to (3-2)-(1-0)=0 yeilds 0.

2. More dominoes

A domino is a rectangle made up of two squares. Each square has 0 to 6 dots on it. A set contains one of each kind of domino.

a) How many dominoes are there in a set?

b) All the dominoes in a set are lined up in a chain (so that the number of dots on the ends of side-by-side dominoes match). If the square at one end has 5 dots on it, how many dots does the square on the other end have?

c) If we take out all the dominoes that have a square with no dot from a set, can we make a chain with the left over dominoes?

Solution:

a) The difference between this question and others like it is that we can have “double 0”, “double 1”, etc. Thus, the answer is $\frac{7\times 6}{2}+7=28$.

b) This first 5 has not been matched up, as all other halves have. So there are seven 5s remaining in the full set (including two on one domino). As this is an odd number, one of them is not matched up, and must be on the end. So a 5 is also at the other end.

c) No. Now there are just seven of each symbol in total (counting both sides when a domino has two of the same). If there were an odd number of just two of the symbols, we could put these at the ends, but with more than that, once we've used any symbol seven times, there will be no usable next domino but many more we must use.

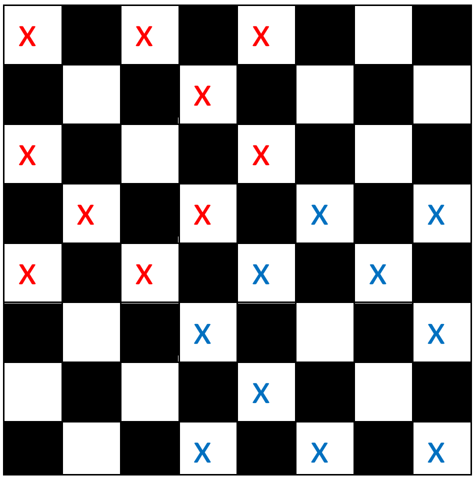
3. Chessboard problems

a) On a chessboard, a knight makes an “L-shaped” move, made up of moving 2 squares in 1 direction, and 1 square in another direction. If a knight starts on one corner square of the board, what is the fewest number of moves it takes for the knight to get to the opposite corner of the board?

b) Is it possible for the knight to get from one corner to the opposite one while landing on every other square on the board exactly once?

c) Is it possible for the knight to be on any square in the board (after starting from the corner) after exactly 6 moves?

d) Is there any number, such that starting from a corner square, the knight can get to any square on the board it likes in exactly that many moves?



Hints and Solutions:

a) *Hint:* Place a night at one corner, and another at the opposite corner of the chessboard. To simplify the problem, the idea is to bring these knights closer by thinking about all the possible first 1-2 and last 1-2 moves of the knight. Where can the knight land after the 1st move? Where can the knight be before the last move? How about after the second move? Mark these squares in red. Or before the second last move? Mark these squares in blue.

What’s the fasted way for a knight to move from a square marked in red to one marked in blue?

Solution: From every red square on the two diagonals closest to the blue squares, you can get to a blue square in 2 moves. No two squares marked by different colours are 1 move apart. So the smallest path will have a total of 6 moves.

For example, in chess notation there is a path of 6 moves starting at a1,b3,d2,c4,e5,f7,h8.

Hence the answer is 6.

b) *Hint:* If the knight has made an even number of moves, can we say anything about the type of square it is on? How about after an odd number of moves?

As the knight starts and ends with white, look at the pattern

White-Black-White-Black-White-...-Black-White.

What do you notice about the numbers of whites and blacks?

Alternatively: What's the number of moves to insure that the knight has landed on all white squares, and has not landed twice on the same square? (the number must be even, and the starting square is excluded from the count!) In the process, how many times did he land on a black square?

Solution: If the knight starts from the white corner, he'll always be on a white square after an even number of moves, and on a black square after an odd number of moves.
The pattern starting with White and ending with White will have 1 less Black than White.
Alternatively: To cover all 31 remaining white squares, he'll need $31 \times 2 = 62$ moves.That covers 31 black squares and we're left with one black square uncovered.

c) *Hint:* 6 is an even number of moves.

Solution: The answer is no. We have already shown that it takes 6 moves, but 6 is even so therefore he can only be on white squares. He cannot land on the blacks after 6 moves.

d) *Solution:* No. If the number is even, then we know he must finish on a white square. If the number is odd, then he must finish on a black square. So after a certain number of moves, there is a colour square the knight is not allowed be on.

Mod 2 Arithmetic

Remember the even/odd rules:

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Sometimes people are too lazy to write "odd" and "even" and so write "1" for all odd numbers and "0" for all even numbers. This is called mod 2 arithmetic, and in one respect it is different from the usual sum rules:

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It's like turning a light switch: if you turn it once, light is on. But if you turn it twice, light is off.

1. Light switch game

Four Christmas light bulbs are arranged in a 2×2 grid. The switches for this grid are so connected that whenever you turn the switch for one of the light bulbs, this also affects the light bulbs immediately up or down, right or left from it. I'd like to know all the possible patterns of light I can get.

a) Examples:

Unlit grid:

0	0
0	0

In the unlit grid, turn the left upper corner light switch: $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 0 \\ \hline \end{array}$

In the unlit grid, turn the right upper corner light switch: $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array}$

In the unlit grid, turn the left lower corner light switch: $\begin{array}{|c|c|}\hline 1 & 0 \\ \hline 1 & 1 \\ \hline \end{array}$

In the unlit grid, turn the right lower corner light switch: $\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & 1 \\ \hline \end{array}$

Now let's turn the switches in succession:

Left Up + Right Up: $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 0 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array} = \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 & 1 \\ \hline \end{array}$

Left Up + Right Up + Right Low:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 0 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 0 & 1 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & 1 \\ \hline \end{array} = \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 0 & 0 \\ \hline \end{array}$$

b) How many patterns are there in total? If you start from the unlit grid, how many patterns can you get by turning switches?

c) If you start from $\begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 0 \\ \hline \end{array}$ how can you get to $\begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 & 0 \\ \hline \end{array}$?

d) If you start from $\begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 0 \\ \hline \end{array}$ how can you get to $\begin{array}{|c|c|}\hline 0 & 0 \\ \hline 0 & 1 \\ \hline \end{array}$?

Solutions:

a) Filled in above.

b) There are $2 \times 2 \times 2 \times 2 = 16$ patterns in total as each light bulb can be either on or off (1 or 0). We can get all 16 patterns:

1 with no lights, 4 with one light each, 2 with one row lighted, 2 with one column lighted, 2 with diagonals lighted and 4 with three lights each.

Encourage the students to work out their diagrams to practice matrix addition mod 2.

By symmetry we may notice that if we turn on two neighbouring (horizontal or vertical) switches, their light bulbs will actually be turned off while the other two will be on.

Two diagonal switches turned one after the other will turn on their own lights and off the other two lights.

Three switches turned in succession will end up lighting 1 light bulb. There are 4 possible combinations of 3 switches, each leading to a different light bulb lit up.

c) *Hint*: How many switches need turning? In terms of matrices, try to solve:

$$\begin{bmatrix}1&0\\0&0\end{bmatrix}+\begin{bmatrix}?&?\\?&?\end{bmatrix}=\begin{bmatrix}0&0\\1&0\end{bmatrix}$$

Solution: Two light-bulbs switch state so in view of above, we'll need two switches turned. The mystery matrix is

$$\begin{bmatrix}1&0\\1&0\end{bmatrix}=\begin{bmatrix}1&1\\0&1\end{bmatrix}+\begin{bmatrix}0&1\\1&1\end{bmatrix}$$

(Upper Right and Lower Right).

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2. Coin game


One face shows the denomination (1 cent, 2 cent, 5 cent, 10 cent). Let's call this face "Cent".
The other face has a 12 star circular border. Let's call this face "Star".

Arrange 16 coins on a table, so that they all have the same face up. One player is turned away from the table, eyes closed. Another player flips any of the coins on the board any number of times, each time saying "Tap", and at the end covers one of the coins with the hand. The first player turns around and has to guess which face of the covered coin is up.
Is there a strategy for guessing all the time?

Solution:

a) Mod 2, the player can count both the number of taps (mod 2) and the number of visible coins with opposite face up (mod 2) by counting 1, 0, 1, 0... If the two numbers are the same, then nothing changed with the hidden coin, if not then it's the opposite face from the beginning of the game.

Said in even-odd language: Number of Taps – Number of visible coins with opposite face up = odd, then the hidden coin is with opposite face up, otherwise it's the same face as at the start of the game.

3. Hat game

Being an extremely cruel and vicious maths teacher, I have decided to take ten of the students and stand them in a line one behind the other. On each of their heads I will place a hat, either black or white, I don't care how many of each colour. Each person will see the colours of all the hats ahead of them, but be unable to see those behind them or their own. The kids starting from the back must tell what colour hat they are wearing, if they pick the

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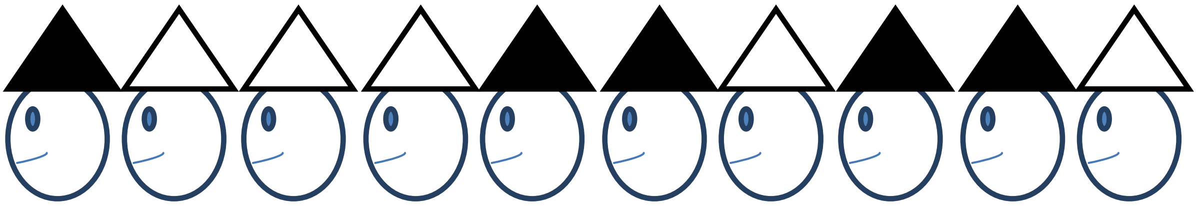

Before we start this whole process, the students are allowed to discuss and to come up with the best strategy in order to save as many as possible…and just so you know you can definitely save at least nine of the kids!!

Hint: This can also be explained with evens and odds, but it seems more fun with 1-s and 0-s. Let's say 1 is black and 0 is white. The person at the end of the row clearly sees which of the other players is a 1 and which is 0, so he's best placed to warn the others, but he is only allowed one number to do it. How can he transmit information about all other players in just one number?

Solution: He tells the sum of all (mod 2). Since the second before last sees all but himself, he can then subtract the sum of the numbers he sees from the total sum he just heard to get his number. Note: $-1 = 1$ (mod 2). It may be worth asking what's -1 (mod 2) before the students start on this problem.

3. Chocolate unwrapping game

http://funschool.kaboose.com/arcade/games/game_chocolate_biz.html

a) Can you describe the game in Mod 2 algebra? How would you represent each move?

b) If you start with a completely wrapped chocolate and then click on each of its squares exactly once, what pattern do you get at the end? Try to answer this without going through all the clicks.

c) If you start with a completely wrapped chocolate and then click on the corner squares and the 4 interior squares, what pattern do you get at the end?

d) How can you win the game in the least possible number of moves?

Solution:

a) This is the same game as with light-bulbs but with 4×4 matrices. The goal of the game is to get from

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

to

1	1	1	1
1	1	1	1
1	1	1	1
1	1	1	1

by adding matrices like these ones:

0	1	0	0
1	1	1	0
0	1	0	0
0	0	0	0

and

1	1	0	0
1	0	0	0
0	0	0	0
0	0	0	0

and

1	1	1	0
0	1	0	0
0	0	0	0
0	0	0	0

etc.

b) *Hint*: Rather than actually go through every click, see how many neighbours (up, down, left or right) are there for each square. Can you guess from here what state that square will be in after all the clicks?

Solution: Each square is affected by 1 when itself, as well as when its neighbours are clicked. If we count the numbers of neighbours of each square together with the square itself:

3	4	4	3
4	5	5	4
4	5	5	4
3	4	4	3

Odd numbers lead to 1 in the end, even numbers lead to 0 in the end:

1	0	0	1
0	1	1	0
0	1	1	0
1	0	0	1

c) There are 16 small squares on the board. The corner switches affect 3 light bulbs each, the other edge switches affect 4 light bulbs each, and the middle switches affect 5 light bulbs each. To divide 16 squares into 4 areas, we could try $16 = 4 + 4 + 4 + 4$ and indeed,