Cool Counting

Notes:

1. You don't have to do all questions from each section in class. You could leave some as homework.

2. We also used this lesson for an introduction to Alcumus (10-15 min)

http://www.artofproblemsolving.com/Alcumus/Introduction.php

- demo sign in (use login: MathsCircles, passwd: Puzzle)

-demo: get problem, solve or give up, get solutions

-Ask students to sign up online and give us their login name next week. There will be monthly challenges and awards for best problem-solvers.

5. Homework: Last questions in the Question sheet (before the extensions), and extensions for the very brave/curious ones.

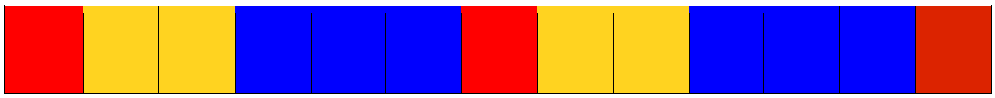
Resources:

-Extra blank paper sheets for those who forgot their note-books!
-Activity sheets
-Possibly calculators.
-Calendars for the extension questions if needed.
-Answer sheets for tutors

Cool Counting

1. A colour pattern

Eoin is painting a border of 2,000 tiles in the following pattern:



And so on. What colour will be the last tile? How many yellow tiles are there in total?

2. How many numbers?

a) How many numbers are there in the list

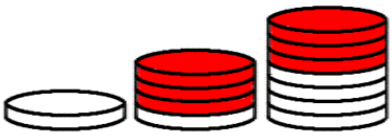
$$37, 38, 39, \ldots, 100?$$

b) How many numbers are there between each of the following pairs of numbers, if we count the beginning and the end too?

$i)\ 1{,}000{,}000\ and\ 2{,}000{,}000 \qquad ii) -50\ and -25 \qquad iii) -102\ \ and\ \ 27$

3. What number comes next?

a) What is the 23nd multiple of 5 in the list $40, 45, 50, 55, \ldots$?
b) What is the 21st number in the list 23, 25, 27, 29, … ?
c) What is the 26th number in the list 25, 28, 31, 34, 37, …?
d) What is the 301st number in the list 2013, 2008, 2003, 1998, …?
e) What is the 50th number in the list 1, 3, 6, 10, 15, 21, …?



4. More lists of numbers

a) How many even numbers are there in the list $22, 23, 24, \ldots, 143, 144$?
b) How many odd numbers are there in the list $22, 23, 24 \ \ldots, 143, 144$?
c) How many numbers in the list $22, 23, 24 \ \ldots, 143, 144$ are divisible by 5?
d) How many numbers in the list above are divisible by neither 2 nor 5?

5. More devious lists of numbers

a) How many 3 digit numbers are divisible by 7?
b) How many 3 digit numbers are perfect squares
(meaning that they are the square of another whole number)?
c) How many numbers are there in the list 27, 38, 49, 60..., 2018?
d) How many numbers are there in the list
1, 2, 5, 10, 17, 26, 37, 50, ..., 485?

Extension Questions

1. How many numbers are there such that if you multiply the number by 3 or if you divide the number by 3, you will have a 3-digit whole number?

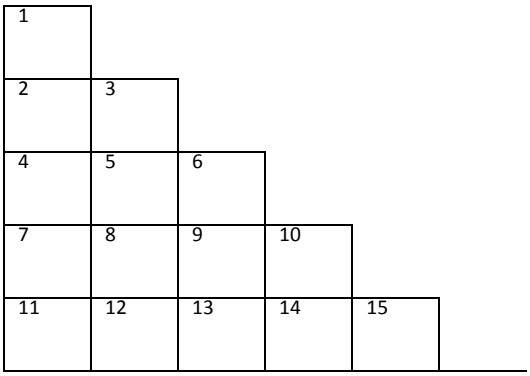
2. How many times does the digit 5 occur in the numbers 1 to 999?

3. How many of the numbers from 1 to 999 have at least one digit of 5, e.g. 315 and 524?

4. Consider the number $123456 \ldots 998999$, which is formed by writing the numbers 1,2,3,4,…,999 in order. What is the 2012th digit from the left?

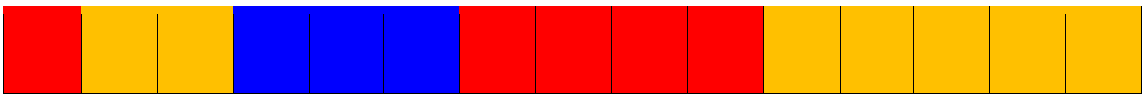
5. What is the most amount of Sundays you can have in 1 month? What about in 1 year?

6. June 17th 2012 was a Sunday. What was the day of the week on 17th June, 1998?
   What will the date be 1 million minutes after noon 1st January 2012?



7. The positive integers are arranged in the pattern indicated in the diagram. What number will be found in the square for the 61st (horizontal) row and 23rd (vertical) column?

8. Eoin is painting a border of 5,050 tiles in the following pattern:

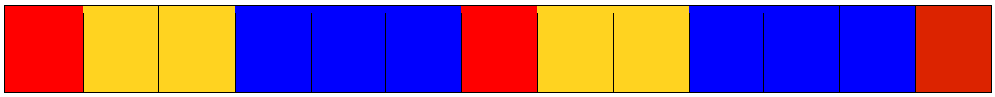


And so on. What colour will be the last tile?
How many blue tiles are there in total?
How many yellow tiles are there in total?
How many red tiles are there in total?

Cool Counting Solutions

1. A colour pattern

Eoin is painting a border of 2,000 tiles in the following pattern:



And so on. What colour will be the last tile? How many yellow tiles are there in total?

Solution:

The pattern repeats every 6 tiles. $2000 \div 6 = 333\ remainder\ 2$ so the last tiles in the pattern will be

There are $333 \times 2 + 1 = 667$ yellow tiles.

2. How many numbers?

a) How many numbers are there in the list

$$37, 38, 39, \dots, 100?$$

b) How many numbers are there between each of the following pairs of numbers, if we count the beginning and the end too?

$i)\ 25\ and\ 50 \qquad ii)\ 34\ and\ 81 \qquad iii)\ 58\ and\ 102 \qquad iv) 1{,}000{,}000\ and\ 2{,}000{,}000$

Hint: Some may say 63 and some may say 64, but which answer is right? Let's start small. This is almost always a very good problem solving technique. How many numbers are there from 1 to 10? How many numbers are there from 0 to 9? From these two examples can we figure out some general rule for how many consecutive numbers there are from one number to another? Can we explain it?

Solutions:

In general, our rule is to subtract the smaller number from the larger and add 1. This is because if we subtract the smaller number from the bigger, it is the same as counting all the numbers between them *except the smaller number.* Hence we must add 1 to compensate for the smaller number.

a) The answer is $100 - 37 + 1 = 64$

This can be explained more clearly using a number line:

1, 2, 3, … , 36, 37, 38, 39, 100

There are 100 numbers above, the first 36 being blue, there remain $100 - 36 = 64$ red ones.

Conclusion: in a list $a, a+1, a+2, \dots, b$ of consecutive numbers there are exactly $b - a + 1$ numbers.

b) $i) 1{,}000{,}001 = 2{,}000{,}000 - 999{,}999 = 2{,}000{,}000 - 1{,}000{,}000 + 1.$

$ii) 26.$ This is the same problem as counting the numbers in the list from 25 to 50.

$iii)\ 27 - (-102) + 1 = 27 + 102 + 1 = 130.$ If in doubt, split the list into

$$-102, -101, -100, \dots, -1 + 0 + 1 + 2 + \cdots + 27$$

There are 102 negative numbers, 27 positive numbers and the number 0. So 130 numbers in total.

3. What number comes next?

a) What is the 23rd multiple of 5 in the list $40, 45, 50, 55, \dots$?

b) What is the 21st number in the list 23, 25, 27,… ?

c) What is the 26th number in the list 25, 28, 31, 34, 37, ...?

d) What is the 301st number in the list 2013, 2008, 2003, 1998, ...?

e) What is the 50th number in the list 1, 3, 6, 10, 15, 21, ...?

Hints and Solutions:

a) The multiples of 5 go like

$40 = 8 \times 5, 45 = 9 \times 5, 50 = 10 \times 5, \dots$

Let's make a table with the numbers and their positions in the list

The initial list	$40 = 8 \times 5$	$45 = 9 \times 5$	$50 = 10 \times 5$		?
The numbers $\div 5$	8	9	10		?
Position in the list	1st	2nd	3rd	...	23rd

Compare the last two rows. Do you notice any patterns?

Indeed, by factorising the numbers we have made a new list beginning with $8, 9, 10, \dots$. The difference between these numbers and their position in the list is always $8 - 1 = 7$. The 23rd term of this new list is $23 + 7 = 30$. Alternatively, to get from the 1st number in the list to the 23rd number we need to add 22 numbers: $8 + 22 = 30$.

Indeed, as discovered in the previous exercise, there are 23 numbers in the list 8, 9, ..., 30. So for the original list, the final answer is $30 \times 5 = 150$.

b) The list goes up by 2-s but the numbers in the list are odd. One approach will be to just list the first 21 elements. If the students do this, ask them to look for patterns that would allow them to come up with a formula for the 21st number in the list.

Pattern 1: Using the fact that the numbers differ by 2-s from the previous one, you may ask the students how each number differs from the first number in the list. To go from the 1st position to the 21st position, we need to perform 20 steps of adding a 2.

Number	23	$25 = 23 + 2$	$27 = 23 + 2 \times 2$	$29 = 23 + 2 \times 3$		$63 = 23 + 2 \times 20$
Position	1st	2nd	3rd	4th	...	21st

This is the usual arithmetic sequence formula:

If a sequence starts with a number a and goes up by r at every step, then it's n-th term is

$$a + r \times (n - 1).$$

Pattern 2: Alternatively, some students may try to change the list to one of consecutive numbers. Since the numbers go every 2-s, you want to divide by 2, but to make life easier you may want to subtract 1 first:

Number	23	25	27		63
Number-1	22	24	26		62
Result $\div 2$	11	12	13		31
Position	1^{st}	2^{nd}	3^{rd}	...	21^{st}

The numbers in the last column are worked out from the last row upward. This is good practice for solving equations as completing the last column upwards, we get to do the operations stated in the first column.

Pattern 3: Some students may suggest adding rather than subtracting 1 which is just as well:

Number	23	25	27		63
Number$+1$	24	26	28		64
Result $\div 2$	12	13	14		33
Position	1^{st}	2^{nd}	3^{rd}	...	21^{st}

c) The list goes up by 3

Number	25	$28 = 25 + 3$	$31 = 25 + 6$		$25 + 75 = 100$
Added	3×0	3×1	3×2		3×25
Position	1^{st}	2^{nd}	3^{rd}	...	26^{th}

d) Starting from 2013, we subtract 5 exactly 300 times so we get $2013 - 5 \times 300 = 513$.

Alternatively: The list goes down by 5-s. If we subtract 3 from each number we get a multiple of 5

Number	2013	2008	2003		$510 + 3 = 513$
Number-3	2010	2005	2000		$102 \times 5 = 510$
Result $\div 5$	402	401	400		102
Position	1^{st}	2^{nd}	3^{rd}	...	301^{st}

Compare the last two rows. Do you see any patterns?
Indeed, looking only at last two rows, the sum of the two elements in each column is 403. Based on this we fill in the last column from down-up.

e) The first number is 1. The second number is $1 + 2$. The third number is $1 + 2 + 3$ …
The 50^{th} number is $1 + 2 + \cdots + 50 = \frac{50 \times 51}{2} = 1275$.
Recall that we may pair $1 + 2 + \cdots + 49 + 50 +$
$50 + 49 + \cdots + 2 + 1 = 50 \times 51 = 2550$
After doing the sum column by column.
One can also play with explaining how to multiply by 50 in one's head:
$51 \times 50 = 51 \times 100 \div 2 = 5100 \div 2 = 5000 \div 2 + 100 \div 2 = 2500 + 50 = 2550.$
We'll discuss related examples in the session on Divisibility tricks.

4. More lists of numbers

a) How many even numbers are there in the list $22, 23, 24, \ldots, 143, 144$?
b) How many odd numbers are there in the list $22, 23, 24, \ldots, 143, 144$?
c) How many numbers in the list $22, 23, 24, \ldots, 143, 144$ are divisible by 5?
d) How many numbers in the list above are divisible by neither 2 nor 5?

Solutions:

a) The even numbers in the list are

$$22, 24, \ldots, 142, 144$$

If we divide all of these by 2 we get

$$11, 12, \ldots, 71, 72$$

which we know how to count. Hence there are $72 - 11 + 1 = 62$ such numbers.

If you do

$$(144 - 22 + 1) \div 2 = 61 \; remainder \; 1$$

But you'd have to explain why this happens. A correct explanation is that if you group the numbers $22, 23$, ..., 142 143, 144 into pairs of 2 consecutive numbers, you get 61 pairs of even-odd, but 1 even number is left without a pair.
Most students notice that the list starts and ends with even numbers and deduce the answer 62 from here.

b)There are $(144 - 22 + 1) = 123$ numbers in all, and 62 are even, so 61 are odd.
Alternatively, you can notice directly that the odds are fewer than the evens by 1.

c) We use the same ideas as above in a). The numbers divisible by 5 are $25, 30, \ldots, 140$. We divide them by 5 to get $5, 6, \ldots, 28$. Hence there are $28 - 5 + 1 = 24$ such numbers.

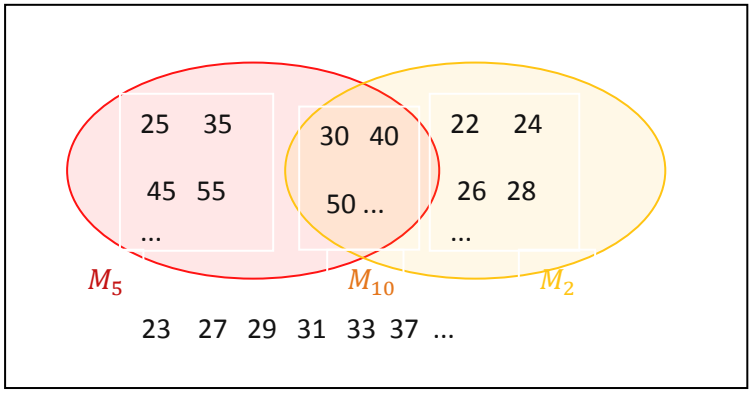
Alternatively, find out how many groups of 5 consecutive numbers there are in the list:

$$(144 - 22 + 1) \div 5 = 123 \div 5 = 24 \; remainder \; 2.$$

Each group contains exactly one number divisible by 5. Among the remaining 2 numbers at the end of the sequence (143, 144) there is no multiple of 5. So there are 24 multiples of 5.

d) We introduce the Ven Diagrams:

Numbers in the list 22, 23, 24, ... 144:



Method I: We already know that there are 61 odd numbers. From these we extract the odd numbers which are multiples of 5. These are 25, 35, 45, ..., 135. They grow by 10 so we write them as

$$25 = 25 + 10 \times 0, 35 = 25 + 10 \times 1, 45 = 25 + 10 \times 2, \dots, 135 = 25 + 10 \times 11.$$

Hence there are 12 such numbers. The final answer is

$$61 - 12 = 49$$

numbers not divisible by 2 nor by 5.

Method II: From the total list of 123 numbers, we can take away the 62 even numbers, and the 24 numbers divisible by 5. But why isn't this the correct answer?

$$123 - 62 - 24 = 123 - 86 = 37$$

It's because we have subtracted twice each of the numbers 30, 40, ...,140 (the multiples of both 5 and 2). There are $14 - 3 + 1 = 12$ such numbers. Putting them back in the sum:

$$37 + 12 = 49.$$

5. More devious lists of numbers

a) How many 3 digit numbers are divisible by 7?
b) How many 3 digit numbers are perfect squares
(meaning that they are the square of another whole number)?
c) How many numbers are there in the list 27, 38, 49, 60..., 2018?
d) How many numbers are there in the list
1, 2, 5, 10, 17, 26, 37, 50, ..., 485?

a) *Solution 1*: The first 3 digit number is 100, but that's not divisible by 7. Some students may check that 101, 102, 103, 104 are not divisible by 7 while 105 is. Is there a faster way to find this? Indeed, $100 \div 7 = 14\ remainder\ 2$, so the first 3 digit number divisible by 7 is $15 \times 7 = 105$. Alternatively, based on the remainder above: $105 = 100 + 5$.
Similarly, $999 \div 7 = 142\ remainder\ 5$, so the last 3 digit number divisible by 7 is $142 \times 7 = 994$. Alternatively, based on the remainder above: $994 = 999 - 5$.
Now we're counting all multiples of 7 in the list 105, ..., 994, or in other words, as seen above, all the consecutive numbers in the list 15, ..., 142.
There are $142 - 15 + 1 = 128$ such numbers.
Solution 2: In the list of three digit numbers there are

$$(999 - 100 + 1) \div 7 = 900 \div 7 = 128, \quad remainder\ 4$$

groups of 7 consecutive numbers. The first 4 numbers in the sequence contain no multiple of 7, so the answer is 128.

b) The first 3 digit number which is a perfect square is $100 = 10^2$. At the other end, we may estimate that $30^2 = 900$ is pretty close to 999, and after some trial-and-error with 31 and 32 we find that $31^2 = 961$ is the largest 3 digit perfect square. The list of consecutive numbers 10, 11, ..., 31 contains 22 numbers.

c) The numbers in this list go by 11-s. Subtracting 27 we get
0, 11, 22, ... 1991. Dividing by 11 we get: 0, 1, 2, ..., 181. There are 182 numbers in this list.

c) The trick is to notice that subtracting 1 we get 0, 4, 9, 16, 25, 36, 49, ..., 484 which is

$$0^2, 1^2, 2^2, 3^2, 4^2, \dots, 22^2$$

So there are 23 numbers in the list.

Solutions to Extension Questions

1. How many numbers are there such that if you multiply the number by 3 or if you divide the number by 3, you will have a 3-digit whole number?

Solution:
If I multiply the number by 3, it must be a 3 digit number. Hence, the number multiplied by 3 must be less than or equal to 999. This means that our number must be less than or equal to 333. Similarly, our number divided by 3 is a 3 digit number. So our number divided by 3 must be greater than or equal to 100. So our number must be greater than 300. This means that our number must be between 300 and 333, and must be divisible by 3. There are 12 of these numbers.

2. How many times does the digit 5 occur in the numbers 1 to 999?

Hint: You may want to count how many times 5 occurs as a unit, then as a ten, then as a hundred digit.
Solution:
As a unit, 5 occurs in 5, 15, 25, ..., 995. By the methods discussed above, we find

$$(995 - 5) \div 10 + 1 = 100 \text{ times.}$$

Alternatively, since we may fill in the first two digits in __ __ 5, we have $10 \times 10 = 100$ numbers.
As a tens digit, 5 occurs in 50, 51, 52, ..., 59 then again in 150, 151,...,159 and so on until 950, 951,...,959. These are 10 groups of 10 numbers, so a total of 100 times.
As a hundreds digit, it occurs in 500, 501, ..., 599. Again, 100 numbers.
All in all, it occurs 300 times.

3. How many of the numbers from 1 to 999 have at least a digit of 5, e.g. 315 and 524?

Hint: How is this related to the previous question?

Solution:
In the previous question, some numbers were counted twice or three times as they contained two or three 5-s:
55, 155, 255, ..., 955 as well as 505, 515, ..., 595 and 550, 551, ..., 559. Note that 555 occurs in all 3 of these lists. So after excluding 555, there are $9 \times 3 = 27$ numbers which contain

two 5-s and 1 number which contains three 5-s. The 27 numbers were counted twice in the previous question, and 555 was counted 3 times. We adjust for these:

$$300 - 27 - 2 = 271.$$

4. Consider the number $123456 \ldots 998999$, which is formed by writing the numbers 1,2,3,4,…,999 in order. What is the 2012th digit from the left?

Hint: In the list 1, 2, ..., 999, split the numbers into 3 categories according to how many digits they have. How many digits does each category contribute to the final number?

Solution:

Numbers grouped by digits	How many	Contributions to the digits of 12345...999
1 digit numbers: 1, ..., 9	9	9 digits
2 digit numbers: 10, ..., 99	90	$90 \times 2 =$180 digits
3 digit numbers: 100,...,999	900	$900 \times 3 =$2700 digits

We see that the 2012th digit falls in the last category. First, we subtract the digits taken up by single and double digit numbers.

$$2012 - (180 + 9) = 1823.$$

How many numbers in the list 100, ..., 999 make up 1823 digits?

$$1823 \div 3 =607 \text{ remainder } 2$$

What is the 607-th number in the list 100, 101, ..., 999?

$$100 + 607 - 1 = 706.$$

We need to consider 2 more digits because of the remainder: 70. So 0 is the 2012th digit.

5. What is the most amount of Sundays you can have in 1 month? What about in 1 year?

Solution: The most days a month can have in 31, that is 4 weeks and 3 days. So the answer is 5, which can be achieved if Sunday is the first, second, or third day of the month. The highest number of days a year can have is 366, which is 52 weeks and 2 days. So if January 1st is a Sunday, Sunday will occur 53 times during the year.

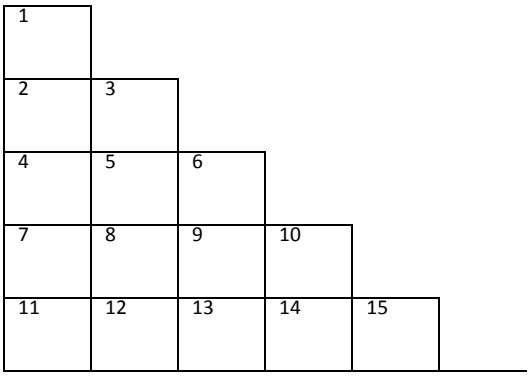
6. June 17th 2012 was a Sunday. What was the day of the week on 17th June, 1998? What will the date be 1 million minutes after noon 1st January 2012?

Hint: If you go behind one non-leap year in the calendar, how will the day of the week on which June 17th falls shift?

Solution: A year usually has 365 days which is 52 weeks and 1 day. So if we go ahead a year, the day of a particular date will shift ahead by 1. Conversely, if we go back a year, the day of a certain date will shift back by 1. However care must be taken with leap years as the extra

day means we shift by 2. In moving back from 2012 to 1998 there are 14 years, 3 of which are leap years: 2012, 2008, 2004, 2000.
So we must shift back by 14+4=18 days. This is the same as shifting back by 4, as 14 days is two weeks. This means that June 17th 1998 is a Wednesday.

For the next problem, 1 day has $24 \times 60 = 1440$ minutes. 1,000,000 = 694x1440+640. In other words, 1 million minutes is the same as 694 days and 640 minutes.
640 minutes is less than 11 hours, so we do not have to worry about it as it won't change the date. $694 = 2 \times 365 - 36$. So if we work back 36 days from January 1st, we'll have our date. December has 31 days and November has 30. So our date is 25th November.



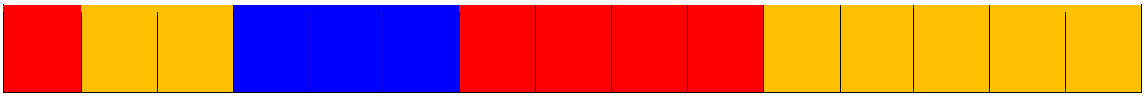
7. The positive integers are arranged in the pattern indicated in the diagram. What number will be found in the square for the 61st (horizontal) row and 23rd (vertical) column?

Solution: To count the numbers on all the rows up to the 60th row:

$$1 + 2 + 3 + \cdots + 60 = 1830.$$

This is the last number on the 60th row.
Then to get the desired number on the 61st row, we simply add 23. So our final answer is 1855.

8. Eoin is painting a border of 5,050 tiles in the following pattern:



And so on. What colour will be the last tile?
How many blue tiles are there in total?
How many yellow tiles are there in total?
How many red tiles are there in total?

Solution:

The number of tiles in the pattern is $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ for some unknown n. This n represents the number of one-colour stripes in the pattern. (A stripe is made of all consecutive tiles of the same colour).
Now $5050 = 101 \times 100 \div 2$ so our $n = 100$. The last tile will be red as the 100 stripes go by the pattern red, yellow, blue, red, yellow, blue ... and $100 \div 3 = 33$ remainder 1.
The number of blue tiles is $3 + 6 + 9 + \cdots + 99$ as 99 is the last multiple of 3.

$$3 + 6 + 9 + \cdots + 99 = 3(1 + 2 + 3 + \ldots + 33) = 1683.$$

There are 33 blue stripes and 33 yellow stripes, and each yellow stripe is 1 tile less than the following blue stripe. So there are

$$2 + 5 + 8 + \cdots + 98 = 1683 - 33 = 1650$$

yellow stripes, and the remaining $5050 - 1650 - 1683 = 1717$.