Ready, Set,…

Summary:

This lesson is based on the game Set. While playing the game, the students learn how to classify and associate objects according to various features. The second part is a lesson in counting based on the game cards. Students may find some parts difficult, in particular when they count the same composite object (pair, set) multiple times and then they need to adjust their answer dividing by the number of times they counted the same object. This is a useful exercise for them, but whenever possible we also provided two different solutions to some of the exercises. It is useful for students to see both solutions. For example, the multiple counting will be something they will encounter later on in combinatorics.

Lesson plan:

1. The Set game: - rules and practice (10 min)

   - play 2-3 rounds (30-35 min)

2. Break (5-10 min)

3. Work in teams on the question sheet. (35 min)

Resources:

1. Decks of Set, one for every group of 5-6 students.

2. One question sheet per student.

3. One Game Rules sheet per table, and/or powerpoint intro.

4. Solutions sheet for the tutors

Aims for instructors:

1. Get students to work together from the start and have fun .

2. Promote basic problem solving strategies:

   - play with examples; here they can use the cards in the deck.

   - start small. Example: choose the 1st card (how many options), then the 2nd etc.

   - organize data in tables, groups, diagrams.

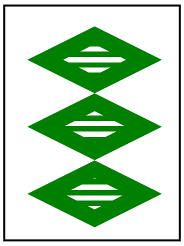
3. Promote student participation: commend good strategies, good starts, good solutions. Call the students by name and explain/let them explain their ideas to the entire class.

The Set Game

Game Rules

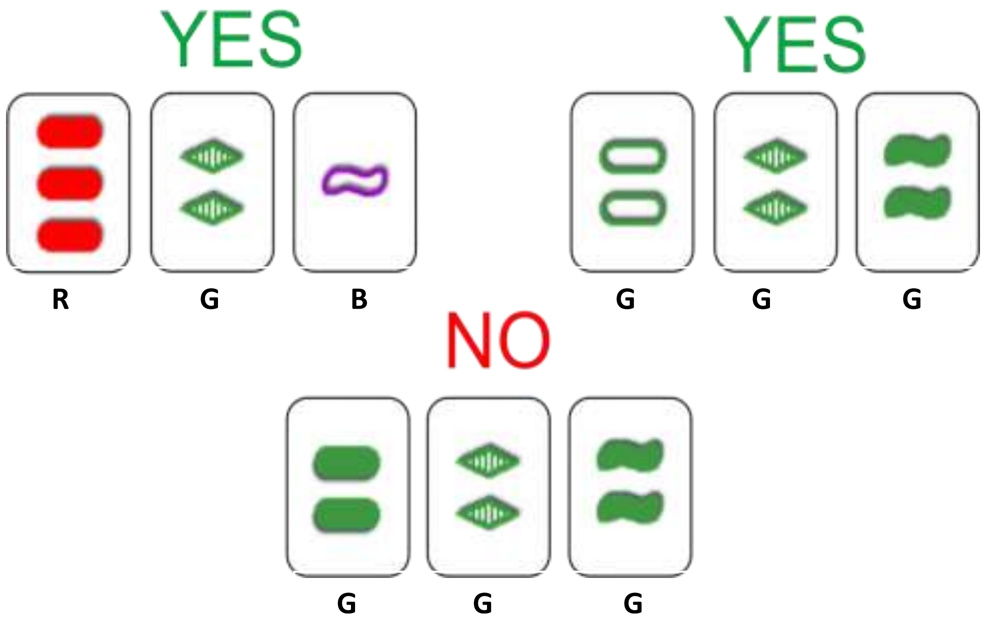
The game is based on a deck of cards varying in 4 features:

Shape (diamond, squiggle, oval)
Number (one, two, or three)
Shading (full, striped, or empty)
Colour (red R, green G, or blue B).



For each possible combination of shape, number, shading, colour, there is exactly one card of that type in the deck.
At all times, 12 cards are placed face up in the middle of the game area. The players have to spot SETS. If you spot one, say SET and show the three cards. If you're right, keep the set. The middle area has to be replenished. If no set can be found, three more cards can be laid down. The winner is the person with most sets at the end of the game.

A SET is made of 3 cards in which each individual feature either stays the SAME on all 3 cards card... OR DIFFERS in each of the 3 cards:



The first example is a set because all features are different: 3 different numbers, 3 different shapes, 3 different shadings, 3 different colours.

The second is a set because there are the same numbers and colours on all cards, but different shapes and shadings. The last is not a set because there are two solid and one striped cards.

Set Game Questions:

1. Here we will explore how many cards of some fixed types there are in the deck.

a) How many cards in the deck contain exactly 1 red diamond?

b) How many cards in the deck contain exactly 1 diamond?

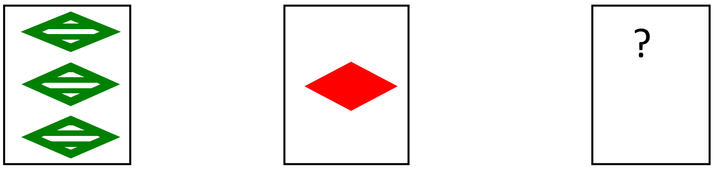
c) How many cards in the deck contain diamonds?

d) How many cards are there in the deck?

2. How many different pairs of cards can be made with the cards in the deck?

3. Choose any pair of cards. In how many ways can you complete it to a SET?

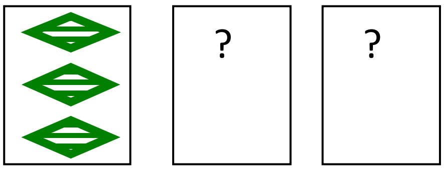
Example:



4. How many SETS can be formed with the cards in the deck?

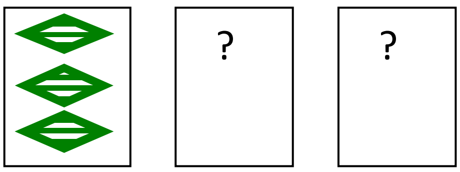
5. A simple strategy is when you decide in advance what to look for in each of the 4 features: cards which are the same or cards which are different.

First, let's try to find the number of sets that have the **same shape, same colour, same shading, but different numbers**. Pick a card from the deck. In how many ways can you complete this to sets that will have the **same shape, same colour, same shading, but different numbers**?

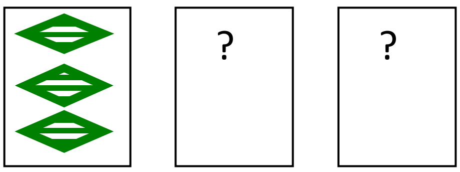


Now imagine you can start with any card. Can you now count the **total** number of sets that have the same shape, same colour, same shading, but different number?

Now repeat this process for different kinds of sets. For instance, in how many ways can you complete the card below to sets that have the **same shape, same colour, but different shading and different numbers**:

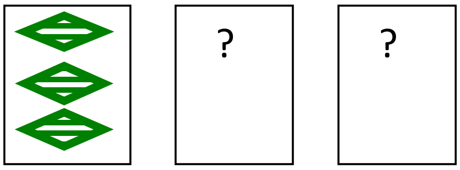


Any more?



Now imagine you can start with any card. Can you now count the **total** number of sets that have the same shape, same colour, but different shading, and different numbers?

Now repeat the process for sets that have the **same shape, different colours, different shadings and different numbers**:



…

In the table below, each row represents a simple strategy. Can you now fill it in using the results above?

Strategy name	Shape	Colour	Shading	Number	How many SETS of this type?
SSSD	Same	Same	Same	Different	
SSDD	Same	Same	Different	Different	
SDDD	Same	Different	Different	Different	
DDDD	Different	Different	Different	Different	
SSSS	Same	Same	Same	Same	

What other simple strategies are there?

Type of strategy	List all strategies of this type
1 feature different, 3 the same	SSSD, SSDS,
2 features different, 2 the same	SSDD, SDDS,
3 features different, 1 the same	SDDD,
4 features different	
Total number of strategies	

Now can you add up your previous answers to find the total number of sets where 1 feature is different and 3 are the same?

6. Which types of SETS are most frequent in the game?

Type of SETS	How many	What percentage
Sets whose cards differ in exactly 1 feature		
Sets whose cards differ in exactly 2 features		
Sets whose cards differ in exactly 3 features		
Sets whose cards differ in exactly 4 features		

7. a) What are the chances that 3 cards chosen randomly from the deck will not be a SET?

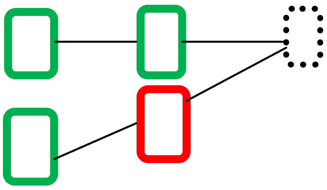
b) What are the chances that 4 cards chosen randomly from the deck will not contain a SET?

Set Game Extension Questions

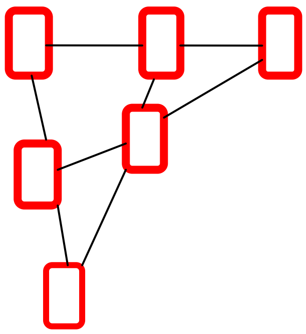
8. In the last round of 12 cards in the game, the very last 3 cards are placed **face down** in the playing area. How can you know with certainty if the three hidden cards form a SET? (No memorization, no guessing, no cheating are necessary).

We will come back to this question in a later session.

9. a) Place any three cards in the green areas. In how many ways can you fill in the red card area so that any three cards connected by a straight line should form a set?



b) Is it possible to place cards in all the red areas so that all 4 lines represent SETS?



Hint: try each feature at a time. For example: In how many ways can you complete the diagram above with 1, 2, 3-s so that they are either all the same, or all different on each line? What happens when you consider all features at the same time?

10. If 5 cards are chosen randomly from the deck, what are the chances that they will not contain a SET?

Set Game Solutions:

1. a) How many cards in the deck contain exactly one red diamond ?

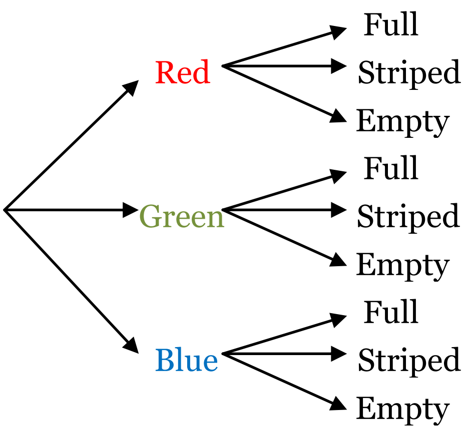
Answer: 3 (solid, striped, or open)

b) How many cards in the deck contain exactly one diamond ?

Answer: $3 \times 3 = 9$. For each of the three colors, there are 3 options of shading.

Full, Red	Striped, Red	Empty, Red
Full, Green	Striped, Green	Empty, Green
Full, Blue	Striped, Blue	Empty, Blue

Student may like tree diagrams better:



c) How many cards in the deck contain diamonds only?

Answer: $3 \times 3 \times 3 = 27$.

d) How many cards are there in the deck?

Answer: $3 \times 3 \times 3 \times 3 = 81$.

2. How many different pairs of cards can be made with the cards in the deck?

Solution 1: $\frac{81 \times 80}{2} = 3240$. You have 81 options when picking the first card, and 80 remaining options when picking the second card, so you have 81×80 ways of getting two cards in order. But in this way we have counted each pair twice: if we call the first card A and the second card B, then the pair AB can also be obtained in reverse order: BA.

Solution 2: Some students may count all pairs made with one card in the deck, then discard that card and look at all pairs with the next card in the deck, etc, so they get: 80+79+...+2+1. It's worth comparing the two methods. For example, note that writing the sum twice and adding columns gives

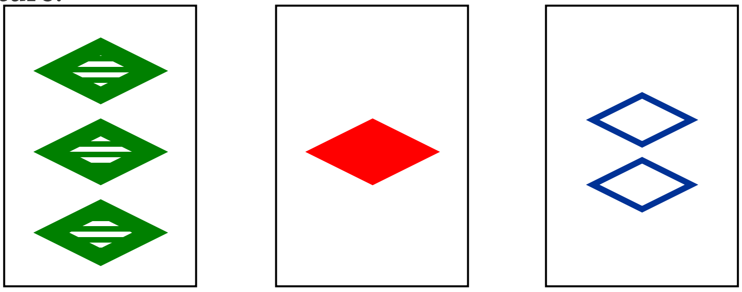
$$\begin{array}{rl} + & 80 + 79 + 78 + \cdots + \ 3 \ + \ 2 \ + \ 1 \\ & 1 + \ 2 \ + \ 3 \ + \cdots + 78 + 79 + 80 \\ & = 81 \times 80 \end{array}$$

So $80 + 79 + 78 + \cdots + \ 3 \ + \ 2 \ + \ 1 = \ 81 \times \frac{80}{2} = 3240$.

3. For each pair of cards, there is exactly ONE card you can add to make a set.

Indeed, looking at the two cards in the pair you find out, for each of the 4 features, whether the cards are going to be all the same or all different. Either way, you have a unique choice for that feature.

In the given example:



4. How many possible sets can be formed with the cards in the deck?

Answer: Any of the 3240 pairs of cards can be completed to a set.

However, each SET contains 3 pairs, so it could be started by 3 different pairs.

So the same set was counting 3 different times, depending on the pair you started with. So there are $\frac{3240}{3}$ =1080 sets.

5. A simple strategy is when you decide in advance what to look for in each of the 4 features: cards which are the same or cards which are different. For example, this is a simple strategy: Find sets where all cards have the **same shape**, the **same colour**, **different shadings**, **different numbers**.

In the table below, each row represents a simple strategy.

Strategy name	Shape	Colour	Shading	Number	How many SETS of this type?
SSSD	Same	Same	Same	Different	$\frac{81}{3} = 27$
SSDD	Same	Same	Different	Different	$\frac{81 \times 2}{3} = 54$
SDDD	Same	Different	Different	Different	$\frac{81 \times 4}{3} = 108$
DDDD	Different	Different	Different	Different	$\frac{81 \times 8}{3} = 216$
SSSS	Same	Same	Same	Same	0

SSSD: *Solution 1:* Choose the 1st card: you have 81 options. Then you know precisely what the other two cards will be: they'll have the same shape, colour, shading, and each a different number. However, there are 3 ways to start the same set ABC: you can start it with A, with B, or with C. So we obtain each set 3 times.

Thus there are actually $\frac{81}{3} = 27$ such sets.

Solution 2: This solution avoids repetition: Since the shape, colour, shading, are the same for all cards, it suffices to choose one shape, colour, and shading for all 3 cards. We have 3 choices for shape and 3 for colour and 3 for shading so

$$3 \times 3 \times 3 = 27 \text{ choices in total.}$$

Note that we have no decision to make as regards the numbers on the cards, since we know we have to use all numbers 1,2,3.

SSDD: *Solution 1:* Choose the 1st card: you have 81 options. Then you know precisely what the other shape and colour of the two cards will be. You also know the two remaining shadings and number, but you have an option as to how to combine them. For example: {solid, striped} and {2, 3} can be combined in 4 different ways:

Card 2	Card 3
2 solid	3 striped
3 striped	2 solid
3 solid	2 striped
2 striped	3 solid

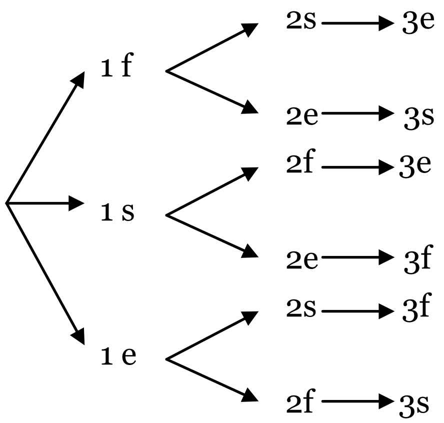
But the first 2 and last 2 rows form the same pair. Thus there are $\frac{81\times2}{3} = 54$ sets.

Solution 2: This solution avoids repetition: Since the shape, colour, are the same for all cards, we first choose one shape and one colour for all 3 cards on our set. We have

$$3 \times 3 = 9 \text{ ways to do that.}$$

Then we'll use all shadings: f, s, e (full, striped, empty) and all numbers 1,2,3. It remains to see in how many ways we can match different shadings with different numbers. You have 3 choices of shadings for 1 (f, s, e), but for each such choice, you have only two remaining options of shadings for 2, and only one remaining for 3. For example if you chose 1 full, then you can't choose 2 full, so you can either have 2 solid or 2 empty. If you chose 2 solid, then 3 has to be empty.

Here's a tree diagram of which shading we can associate to which number:



So we have 6 ways to match shadings to numbers.

All in all we have $3 \times 3 \times 6 = 54$ sets.

SDDD: *Solution 1:* There are $\frac{81\times4}{3} = 108$ such sets. For cards 2 and 3, check that you can combine 2 colours, 2 shadings and 2 numbers to form 4 different pairs.

Solution 2: Since all the cards have the same shape, you can choose the shape on the set in 3 ways. Then you will use all possible colours, shadings and numbers, but have to decide how to match them.
There are 6 ways to match shadings with numbers (as above).
There are 6 ways to match colours with numbers (similar to above).
Now you have completely constructed your set. You get

$$3 \times 6 \times 6 = 108 \text{ sets.}$$

DDDD: *Solution 1:* There are $\frac{81\times8}{3} = 216$ such sets. For cards 2 and 3, check that you can combine 2 shapes, 2 colours, 2 shadings and 2 numbers to form 8 different pairs.
SSSS: 0 sets. you can't have 3 identical cards!

Solution 2: You have to match all the shapes, the colours and the shadings with the numbers 1,2,3. You have

$$6 \times 6 \times 6 \text{ ways to do this}$$

What other simple strategies are there?

Type of strategy	List all strategies of this type
1 feature different, 3 the same	SSSD, SSDS, SDSS, DSSS
2 features different, 2 the same	SSDD, SDSD, SDDS, DSSD, DSDS, DDSS
3 features different, 1 the same	SDDD, DSDD, DDSD, DDDS
4 features different	DDDD
Total number of strategies	$15 = 2^4 - 1$

Answer: There are 15 possible strategies. Each feature could be either the same for all cards or different for all cards: 2 options per feature. For numbers, shape, shadings, colors, we have $2 \times 2 \times 2 \times 2$=16 options. However it's not possible that all features are the same at the same time, because there are no three cards of the same type. So 15 possible strategies.

6. Which types of SETS are most frequent in the game?

Type of SETS	How many	What percentage
Sets whose cards differ in exactly 1 feature	$27 \times 4 = 108$	$\frac{108}{1080} = 10\%$
Sets whose cards differ in exactly 2 features	$54 \times 6 = 324$	$\frac{324}{1080} = 30\%$
Sets whose cards differ in exactly 3 features	$108 \times 4 = 432$	$\frac{432}{1080} = 40\%$
Sets whose cards differ in exactly 4 features	216	$\frac{216}{1080} = 20\%$

7. a) What are the chances that 3 cards chosen randomly from the deck will not be a SET?

Answer: Choose the first 2 cards at random. There are $81 - 2 = 79$ cards left in the deck. Among these, there's 1 card which can form a SET with the first two. So the chances of not getting a set are $\frac{78}{79} \cong 98.73\%$

b)What are the chances that 4 cards chosen randomly from the deck will not contain a SET?

Answer: Choose the first 2 cards at random.

You have $\mathbf{\frac{78}{79}}$ chances of not getting a set when choosing the 3rd card.

Now you have 3 cards A, B, C, you can form 3 pairs: AB, AC or BC. When you choose the 4th card, you must be careful not to complete any of the 3 pairs to a set. Each pair requires a different card to form a set, so 3 cards must be avoided.

You have $\mathbf{81 - 3 = 78}$ options left for the 4th card. $\mathbf{78 - 3 = 75}$ of them don't form a set with any of AB, AC, BC.

In total after choosing the 3rd and 4th card, you have $\frac{78}{79} \times \frac{75}{78} = \frac{75}{79} \cong 96.15\%$ chances of not forming a SET with 4 random cards.

Extension Questions:

8. In the last round of 12 cards in the game, the very last 3 cards are placed **face down** in the playing area. How can you know with certainty if the three hidden cards form a SET?

Answer: Fix a feature: shape. In a SET, the numbers of cards of each kind can distributed in 4 ways:

Diamonds	Ovals	Squiggles
3	0	0
0	3	0
0	0	3
1	1	1

In all these cases, the differences

Diamonds – Ovals, Ovals – Squiggles, Diamonds – Squiggles are either 0, 3 or -3.

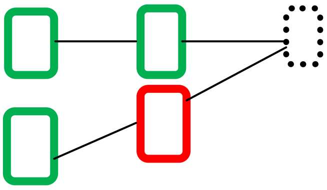
At the beginning there are as many diamonds as ovals and as squiggles in the deck. So after removing some sets, the remaining cards should always satisfy:

Diamonds – Ovals is a multiple of 3.

Ovals – Squiggles is a multiple of 3.

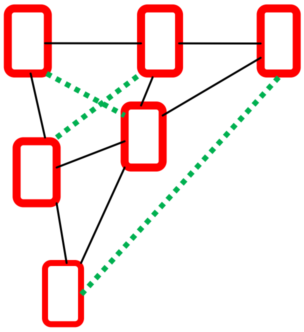
Diamonds – Squiggles is a multiple of 3.

9. a) Place any three cards in the green areas. In how many ways can you fill in the red card area so that any three cards connected by a straight line should for a set?



Answer: You have only one way of filling in the dotted place so that the first row makes a SET. Then you have 2 known cards on the 2nd row and only one way to complete it to a SET by choosing the card in the red place.

b) Is it possible to place cards in all the red areas so that all 4 black lines represent SETS?

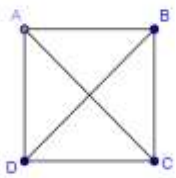


Hint: try each feature at a time. For example: In how many ways can you complete the diagram above with 1, 2, 3-s so that they are either all the same, or all different on each line? What happens when you consider all features at the same time?

Answer: In all the solutions you can find, the cards in each pair connected by a green dashed line will always have to have the same features, so all in all they'll have to be identical. Impossible.

10. If 5 cards are chosen randomly from the deck, what are the chances that they will not contain a SET?

Answer: We saw in Question 7 that there are $\frac{78}{79} \times \frac{75}{78} = \frac{75}{79}$ chances of not forming a SET when picking 4 random cards. Once you have 4 cards, you can form 6 pairs:



Of these, it is possible that **either**

a) AB and CD will need the same 3rd card to make a set, **or**

b) AD and BC will need the same 3rd card to make a set, **or**

c) AC and BD will need the same 3rd card to make a set.

So the $\frac{75}{79}$ chances to choose 4 points without forming a SET split into:

- $\frac{3}{79}$ ways in which either a) or b) or c) can happen. Then there are $\frac{77-5}{77} = \frac{72}{77}$ ways to choose the 5th card without forming a set with any of the 6 pairs. (Remember, in this case 2 of the pairs will form the same set, hence the 5 at the numerator. After choosing 4 cards you have $81 - 4 = 77$ cards left).
- $\frac{72}{79}$ ways in which neither a) or b) or c) can happen. Then there are $\frac{77-6}{77} = \frac{71}{77}$ ways to choose the 5th card without forming a set with any of the 6 pairs.

Chances of not forming a SET from among 5 randomly chosen cards:

$$\frac{3}{79} \times \frac{72}{77} + \frac{72}{79} \times \frac{71}{77} \cong 87.58$$

The story behind the game:

The game was invented by Population geneticist Marsha Falco when trying to connect the traits of dogs to their genes. To help herself understand what she was looking at, she put the information about each dog on file cards. She used symbols with different Features to indicate different genes. While explaining what to look for in the file cards to the veterinarians around, she came up with the idea for the Set game.