Distributivity and related number tricks.

Notes:

No calculators are to be used.

Each group of exercises is preceded by a short discussion of the concepts involved and one or two examples to be worked out with the class before students start working on their own on that section.

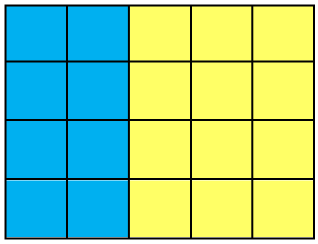
A fairly quick tempo of solutions discussions can be kept during the arithmetic problems.

Not all parts of a question need be solved, some (preferably those similar to ones already solved) can be left as homework.

Resources

- Question sheets – one per student.

Distracting Distributivity



Counting squares:

$$4 \times (2+3) = 4 \times 2 + 4 \times 3.$$

The Distributivity Law: $\boxed{\boldsymbol{a(b+c) = ab + ac = (b+c)a.}}$

1. Mental Arithmetic: $\times 99$ and $\times 999$.

a) Try these for a warm up:

i) $500 - 75$ ii) $800 - 78$ iii) $5{,}000 - 625$

b) Quick multiplication: try these in your head or on paper using distributivity:

i) 75×99 ii) 78×99 iii) 625×999

c) Use a distributivity trick to solve these $\times 18$ and $\times 198$ exercises:

i) 335×18 ii) 128×198 iii) 154×198

d) Solve these as fast as you can without a calculator:

i) $333{,}333 \times 11$ ii) $232{,}323 \times 11$ iii) $999{,}999 \times 7$

2. Difference of two squares: $\boxed{(x-y)(x+y) = x^2 - y^2}$

a) Quick multiplication: Solve these without a calculator or long multiplication:

i) 39×41 ii) 28×32 iii) 67×73

b) Find a square number with the property that adding 101 to it gives you another square number.

c) Find the larger number in each of these pairs:

i) $(\sqrt{19} - 4)(\sqrt{19} + 4)$ or $(\sqrt{7} - \sqrt{3})(\sqrt{7} + \sqrt{3})$? ii) $\frac{1}{\sqrt{50}-7}$ or $\sqrt{50}$+7?

3. More Arithmetic Tricks:

a) Find A, B, C, D, E and F without using any calculators:

$A = 5999 \times 2001 - 2001 \times 2999 + 3000 \times 1999 - 5999.$
$B = 4848 \times 63 + 693 + 63 \times 5252.$
$C = 171 \times 54 - 53 \times 161.$
$D = 1995 \times 1998 - 1997 \times 1996.$
$E = 999 \times 54{,}321 - 45{,}679.$
$F = 2016 \times 2015 - 2015 \times 2014 - 2014 \times 2013 + 2013 \times 2012.$

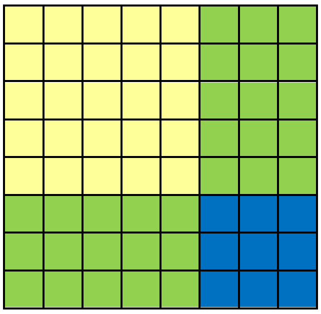
4. Products of negative numbers and distributivity.

a) It's easy to see that $2 \times (-3) = (-3) + (-3) = -6$.
But why is it that $(-2) \times (-3) = 6$?

b) We now know that $-c = (-1) \times c$ and $c = (-1) \times (-c)$ for all numbers c.
Can you also explain why the following properties are true for all numbers a and b:

i) $-(-a) = a.$ ii) $-(a+b) = -a - b.$ iii) $-(a-b) = -a + b$

5. Squares of sums and differences.



Counting squares:

$$8^2 = (5+3)\times(5+3) = 5\times(5+3) + 3\times(5+3)$$
$$= 5^2 + 5\times 3 + 3\times 5 + 3^2$$

In general, for any two unknown numbers x and y:

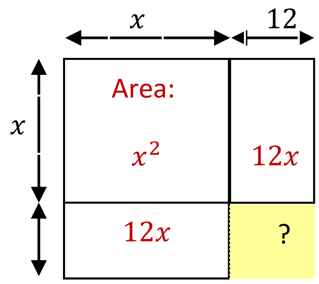
The Square of a Sum formula:

$$\boxed{(x+y)^2 = x^2 + 2xy + y^2.}$$

a) The number 13 has some strange property: $13^2 = 169.$
If you read 13 backwards, and square: $31^2 = 961.$
Find more two-digit numbers with the same property. Explain why it works.

b) Think of a number. Add 10 to it. Multiply your result by the original number. Add 25.
Take the square root. Subtract 5. What do you get? Does it always work the same way? Explain.

c) In each of the following equations, solve for x:



i) $x^2 + 24x = 9856.$
Hint: In the picture to the right, you have a square of side $x + 12$ with a missing corner. Find the area of the missing corner then add it on to both sides of the equation above.

ii) $x(x + 202) = 29{,}799.$

Hint: As above, write the left hand side as the area of a square with a missing corner. Add the missing corner to both sides.

The Square of a Difference formula:

$$\boxed{(x-y)^2 = x^2 - 2xy + y^2.}$$

d) Try to solve these without using your calculator or long multiplication:

$i)\ 19^2 \qquad ii) 49^2 \qquad iii)\ 99^2 \qquad iv)\ 199^2 \qquad v) 299^2$

e) Add a number to $x^2 - 24x$ to get a square of the form $(x-y)^2$. Using this, what is the smallest value that the expression $x^2 - 24x$ can take for random values of x?

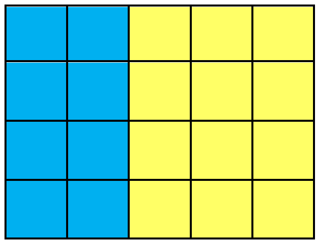
f) The result of each of these calculations is an integer number. Find it in each case:

$i)\ \sqrt{4+2\sqrt{3}} - \sqrt{4-2\sqrt{3}}$ ii) $\left(\sqrt{7}-1\right)^2 + \sqrt{12+2\sqrt{35}} + \sqrt{12-2\sqrt{35}}$

6. A mind-reader computer programme:

Play this game a number of times. Can you explain what's going on?
http://www.cut-the-knot.org/Curriculum/Magic/MindReaderNine.shtml

Distracting Distributivity - Solutions



Counting squares:

$$4 \times 5 = 4 \times (2+3) = 4 \times 2 + 4 \times 3,$$

Working backwards to count the blue squares:

$$4 \times 2 = 4 \times (5-3) = 4 \times 5 - 4 \times 3.$$

$4 \times$ gets distributed to 5 and -3.

Clearly the same counting argument works if we replace 4, 2, 3 by any unknown numbers a, b and c. They may be positive or negative.

The Distributivity Law:

$$\boxed{a(b+c) = ab + ac = (b+c)a.}$$

(When working with letters, it is customary to omit the $\times$ sign, especially when it can be confused with x. So $a(b+c)$ stands for $a \times (b+c)$ while ab stands for $a \times b$).

1. Mental Arithmetic: $\times 99$ and $\times 999$.

With some practice and the Distributivity Law you can get to mentally solve multiplication problems like 68×99 or 637×999.

Examples:

i) $68 \times 99 = 68 \times (100-1) = 68 \times 100 - 68 \times 1 = 6800 - 68 = 6732$ and

ii) $637 \times 999 = 637 \times (1{,}000 - 1) = 637 \times 1{,}000 - 637 \times 1 = 637{,}000 - 637 = 636{,}363.$

Practice exercises:

a) Try these for a warm up:

i) $500 - 75$ ii) $800 - 78$ iii) $5{,}000 - 625$

b) Quick multiplication: try these in your head or on paper using distributivity:

i) 75×99 ii) 78×99 iii) 625×999

c) Use a distributivity trick to solve these $\times 18$ and $\times 198$ exercises:

i) 335×18 ii) 128×198 iii) 154×198

d) Solve these as fast as you can without a calculator:

i) $333{,}333 \times 11$ ii) $232{,}323 \times 11$ iii) $999{,}999 \times 7$

Solutions:

a) i) 425 ii) 722 iii) 4375

b)i) 7425 ii) 7722 iii) 624,375

c) Write 18=20-2 and 198=200-2:

i) $335 \times 18 = 335 \times 20 - 335 \times 2 = 6700 - 670 = 6030.$

ii) $128 \times 198 = 128 \times 200 - 128 \times 2 = 25{,}600 - 256 = 25{,}344.$

iii) $154 \times 198 = 154 \times 200 - 154 \times 2 = 30{,}800 - 308 = 30{,}492.$

d) i) $333{,}333 \times 11 = 3{,}333{,}330 + 333{,}333 = 3{,}666{,}663$ ii) 2,555,553 iii) 6,999,993

2. Difference of two squares.

Example: Let's try 48×52 without long multiplication. The trick is to notice that

$52 = 50 + 2$ while $48 = 50 - 2$ so:

$$48 \times 52 = (50-2) \times (50+2) = (50-2) \times 50 + (50-2) \times 2$$
$$= 50 \times 50 - 2 \times 50 + 50 \times 2 - 2 \times 2 = 2500 - 100 + 100 - 4 = 2496.$$

General trick:

Take any two numbers, call them x and y. Then the product between their difference and their sum is:

$$\boxed{(x-y)(x+y) = x^2 - y^2}$$

Proof: $(x-y)(x+y) = (x-y)x + (x-y)y = xx - yx + xy - yy = x^2 - y^2.$

Example: $59 = 60 - 1$ and $61 = 60 + 1$ so we can apply the difference of squares formula for $x = 60$ and $y = 1$.

$$59 \times 61 = (60-1)(60+1) = 60^2 - 1 = 3600 - 1 = 3599$$

Practice exercises:

a) Quick multiplication:

i) 39×41 ii) 28×32 iii) 67×73

Solutions:

i) $39 \times 41 = (40+1)(40-1) = 40^2 - 1 = 1{,}600 - 1 = 1{,}599$

ii)896 iii)4,891

b) Find a square number with the property that adding 101 to it gives you another square number.

Solution:

Let the unknown square number be x^2. Then $x^2 + 101 = y^2$. Subtract x^2 from both sides to get $y^2 - x^2 = 101$.

$$(y-x)(y+x) = 101.$$

We could try the simultaneous equations
$$y - x = 1$$
$$y + x = 101$$

To get $y = 51$ and $x = 50$.

c) Find the larger number in each of these pairs:

i) $(\sqrt{19}-4)(\sqrt{19}+4)$ or $(\sqrt{7}-\sqrt{3})(\sqrt{7}+\sqrt{3})$? ii) $\frac{1}{\sqrt{50}-7}$ or $\sqrt{50}$+7?

Solution:

i) The first number is $19 - 16 = 3$ while the second is $7 - 3 = 4$, larger.

ii) The two numbers are equal. Indeed, in the first fraction we can multiply both numerator and denominator by $\sqrt{50}$+7 to get:

$$\frac{1}{\sqrt{50}-7} = \frac{\sqrt{50}+7}{(\sqrt{50}-7)(\sqrt{50}+7)} = \frac{\sqrt{50}+7}{50-49} = \sqrt{50} + 7.$$

3. More Arithmetic Tricks:

Sometimes it pays to pay attention to repeating terms in long calculation. For example, if a number a comes up in two different products ab and $-ac$, then writing $ab - ac$ like $a(b-c)$ can simplify things considerably.

Example:

Find A and B without using any calculators or long multiplication:

$A = 3333 \times 6543 + 2222 \times 6667 - 3333 \times 4321.$
$B = 499 \times 1112 - 1111 \times 399.$

Solution:
We notice that 3333 comes up twice in this calculation, so we're going to use distributivity.
Since it doesn't matter in which order we do the additions and subtractions,
A= $3333 \times 6543 - 3333 \times 4321 + 2222 \times 6667$
= $3333 \times (6543 - 4321) + 2222 \times 6667$
= $3333 \times 2222 + 2222 \times 6667$
= $2222 \times (3333 + 6667)$
= $2222 \times 10000 = 2{,}222{,}000.$

No factor repeats in B, but some numbers are just too similar to each other. Let's write $1112 = 1111 + 1$:
$B = 499 \times 1112 - 1111 \times 399$
$= 499 \times (1111 + 1) - 1111 \times 399$
$= 499 \times 1111 + 499 - 1111 \times 399.$ Now we move 499 to be the last in the sum,
$= 499 \times 1111 - 399 \times 1111 + 499$ so that we can use distributivity for 1111.
$= (499 - 399) \times 1111 + 499$
$= 100 \times 1111 + 499$
$= 111{,}100 + 499 = 111{,}599.$

Practice exercises:
a) Find A, B, C, D, E and F without using any calculators:

$A = 5999 \times 2001 - 2001 \times 2999 + 3000 \times 1999 - 5999.$
$B = 4848 \times 63 + 693 + 63 \times 5252.$
$C = 171 \times 54 - 53 \times 161.$
$D = 1995 \times 1998 - 1997 \times 1996.$
$E = 999 \times 54{,}321 - 45{,}679.$
$F = 2016 \times 2015 - 2015 \times 2014 - 2014 \times 2013 + 2013 \times 2012.$

Solutions:
$A = 2001 \times (5999 - 2999) + 3000 \times 1999 - 5999 = 2001 \times 3000 + 3000 \times 1999 - 5999$
$= 3000 \times (2001 + 1999) - 5999 = 3000 \times 4000 - 5999 = 12{,}000{,}000 - 5999$
$= 11{,}994{,}001.$
Or $A = 2{,}000 \times 5{,}999 + 5{,}999 - 2{,}000 \times 2{,}999 - 2{,}999 + 2{,}000 \times 3{,}000 - 3{,}000 - 5{,}999$
$= 2{,}000(5{,}999 - 2{,}999 + 3{,}000) - 2{,}999 - 3{,}000$
$= 2{,}000(6{,}000) - 5{,}999 = 11{,}994{,}001.$
$B = 636{,}993$
$C = 171 \times (53 + 1) - 53 \times 161 = 171 \times 53 + 171 - 53 \times 161 = 53 \times (171 - 161) + 171$
$= 530 + 171 = 701$
$D = 1995 \times (1997 + 1) - 1997 \times 1996 = 1995 \times 1997 + 1995 - 1997 \times 1996 = 1995 - 1997$
$= -2$
$E = 54{,}321 \times (1000 - 1) \times 54{,}321 - 45{,}679 = 54{,}321{,}000 - 54{,}321 - 45{,}679$
$= 54{,}321{,}000 - (54{,}321 + 45{,}679) = 54{,}321{,}000 - 100{,}000 = 54{,}221{,}000.$
$F = 2016 \times 2015 - 2015 \times 2014 - 2014 \times 2013 + 2013 \times 2012 =$
$2015 \times 2 - 2013 \times 2 = 2 \times 2 = 4$

4. Products of negative numbers and distributivity.

a) It's easy to see that $2 \times (-3) = (-3) + (-3) = -6$.
But why is it that $(-2) \times (-3) = 6$?

Answer: Distributivity! Once people noticed the property

$$a(b + c) = ab + ac$$

for all positive numbers, they decided that the negative numbers must abide by it as well. When trying to calculate $(-2) \times (-3)$, they first thought about the meaning of (-2). They remembered that (-2) was defined by the condition

$$(-2) + 2 = 0.$$

From here, all they had to do was put a (-3) into the equation and use distributivity:

$$(-2) \times (-3) + 2 \times (-3) = \big((-2) + 2\big) \times (-3) = 0 \times (-3) = 0.$$

But $2 \times (-3) = -6$ so the equation above says $(-2) \times (-3) - 6 = 0$, that is

$$(-2) \times (-3) = 6.$$

In fact, the same proof works for any positive numbers a and c:

$$(-a) \times (-c) = ac .$$

b) We now know that $-c = (-1) \times c$ and $c = (-1) \times (-c)$ for all numbers c.
Can you also explain why the following properties are true for all numbers a and b:
i) $-(-a) = a$.
ii) $-(a + b) = -a - b$.
iii) $-(a - b) = -a + b$

Answers:

b) i) Both $-(-a)$ and a are solutions of to the question $-a+ \ ? = 0$.

ii) Both $-(a + b)$ and $-a - b$ are solutions of to the question $a + b + \ ? = 0$.

Or, $-(a + b) = (-1) \times (a + b)$ and apply distributivity.

iii) Both $-(a - b)$ and $-a + b$ are solutions of to the question $a - b + \ ? = 0$.

Or, $-(a - b) = (-1) \times (a - b)$ and apply distributivity.

BTW: Long multiplication/division and distributivity.

The Distributivity Law is the reason behind the long multiplication and long division algorithms.
a) Long multiplication for 23 × 761:
First 761 is split from right to left: 761 =1+60+700. Then by distributivity:

23 × 761 = 23 × 1 + 23 × 60 + 23 × 700
= 23 + 1380 + 16100

The long multiplication is a particular arrangement of the numbers with alignment by the last digit, like this:

```
    2 3
× 7 6 1
-------
    2 3
1 3 8 0
```

1 6 1 0 0

---

1 7 5 0 3

where, in the first row, 23 = 23 × 1, in the second row, 1380 = 23 × 60, and, in the third row, 16100 = 23 × 700. In time, people have tired of writing the trailing zeros that are due to the powers of 10, and now remember them just by the placement of the other digits.
(Copied from : http://www.cut-the-knot.org/Curriculum/Arithmetic/LongMultiplication.shtml
Check the website for a nice Java applet and other resources.)

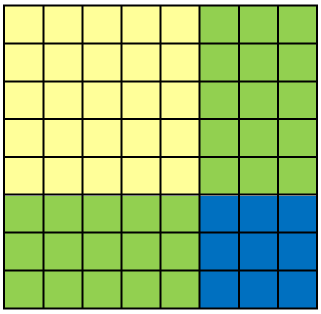
b) Long division is also a consequence of the Distributivity Law:

$$17503 = 16100 + 1380 + 23 = 23 \times 700 + 23 \times 60 + 23 \times 1 = 23 \times 761.$$

Long division splits this into steps by teasing out the hundreds, the tens and then the units:

```
    761
23 | 17503
   161
    140
    138
      23
      23
       0
```

5. Squares of sums.



Counting squares:

$$8^2 = (5+3) \times (5+3) = 5 \times (5+3) + 3 \times (5+3)$$
$$= 5^2 + 5 \times 3 + 3 \times 5 + 3^2$$

In general, for any two unknown numbers x and y:

The Square of Sum formula:

$$\boxed{(x+y)^2 = x^2 + 2xy + y^2.}$$

Example:
Mental arithmetic: $31^2 = (30 + 1)^2 = 900 + 2 \cdot 30 + 1 = 961.$

Practice exercises:
a) The number 13 has some strange property: $13^2 = 169.$
If you read 13 backwards, and square: $31^2 = 961.$
Find all two-digit numbers with the same property.

Answer:

$$\begin{aligned} 11^2 &= 121, & 11^2 &= 121; \\ 12^2 &= 144, & 21^2 &= 441; \\ 22^2 &= 484, & 22^2 &= 484; \\ 10^2 &= 100, & 01^2 &= 001. \end{aligned}$$

In general, a number xy with digits x and y will square like this:
$(10x + y)^2 = 100x + 20xy + y^2$ while the number read backwards squares like
$(10y + x)^2 = 100y + 20xy + x^2$.

The backwards reading trick works as long as x^2, $2xy$ and y^2 are all digits.

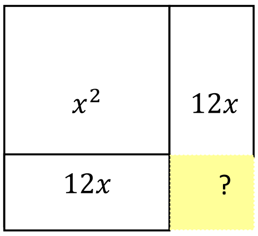
b) Think of a number. Add 10 to it. Multiply your result by the original number. Add 25. Take the square root. Subtract 5. What do you get? Explain.

Answer: Think of a number. x
Add 10 to it. $x + 10$
Multiply your result by the original number. $(x + 10)x$
Add 25. $(x + 10)x + 25 = (x + 5)^2$ so

$\sqrt{(x + 10)x + 25} = x + 5$ or $-(x + 5)$, depending on which of the last two numbers is positive.
So $\sqrt{(x + 10)x + 25} - 5 = x$ if $x + 5 \geq 0$ and $-x - 10$ if $x + 5 < 0$.
To convince the audience that the sign is important, go through the previous steps with $x = -10$ as an example.

c) In each of the following equations, solve for x:

i) $x^2 + 24x = 9856$.
Hint: First find the missing area in the picture:



ii) $x(x + 202) = 29{,}799$.

Answers:
i) The square in the picture is completed by $12^2 = 144$. Add this to both sides of the equation:
$x^2 + 24x + 144 = 9856 + 144 = 10{,}000 = 100^2$ so $(x + 12)^2 = 100^2 = (-100)^2$,
thus $x + 12 = 100$ or -100 and so $x = 88$ or $x = -112$.
ii) $x(x + 202) = x^2 + 202x$ is completed to a square by $101^2 = 10201$. Add to both sides:
$x^2 + 202x + 10201 = 29{,}799 + 10201 = 40{,}000$, so $(x + 101)^2 = 200^2 = (-200)^2$
hence $x = 99$ or $x = -301$.

The Square of Difference formula:

$$\boxed{(x - y)^2 = x^2 - 2xy + y^2.}$$

d) Try to solve these without using your calculator or any long multiplication:

$i)\ 19^2$ $\quad ii)49^2$ $\quad iii)\ 99^2$ $\quad iv)\ 199^2$ $\quad v)299^2$

Answers:
i) $(20 - 1)^2 = 400 - 40 + 1 = 361$; $\quad ii)\ (50 - 1)^2 = 2500 - 100 + 1 = 2401$.

$iii)\ (100 - 1)^2 = 10{,}000 - 200 + 1 = 9801$. $\quad iv)(200 - 1)^2 = 40{,}000 - 400 + 1 = 39{,}601$.

$$v)(300 - 1)^2 = 90{,}000 - 600 + 1 = 89{,}401.$$

e) A square number is always positive or at least zero. Using this, what is the smallest number you can get when you calculate $x^2 - 24x$ for random values of x?

Answer: Complete $y + 144 = x^2 - 24x + 144 = (x - 12)^2 \geq 0$. So $y \geq -144$.

f) The result of each of these calculations is an integer number. Find it in each case:

i) $\sqrt{4 + 2\sqrt{3}} - \sqrt{4 - 2\sqrt{3}}$ ii) $\left(\sqrt{7} - 1\right)^2 + \sqrt{12 + 2\sqrt{35}} + \sqrt{12 - 2\sqrt{35}}$

Answer:

$$i)\ \sqrt{4 + 2\sqrt{3}} - \sqrt{4 - 2\sqrt{3}} = \sqrt{3 + 2\sqrt{3} + 1} - \sqrt{3 - 2\sqrt{3} + 1}$$
$$= \sqrt{\left(\sqrt{3} + 1\right)^2} - \sqrt{\left(\sqrt{3} - 1\right)^2} = \sqrt{3} + 1 - \left(\sqrt{3} - 1\right) = \sqrt{3} + 1 - \sqrt{3} + 1 = 2.$$

ii) $\left(\sqrt{7} - 1\right)^2 + \sqrt{12 + 2\sqrt{35}} + \sqrt{12 - 2\sqrt{35}} = 7 - 2\sqrt{7} + 1 + \sqrt{\left(\sqrt{7} + \sqrt{5}\right)^2} + \sqrt{\left(\sqrt{7} - \sqrt{5}\right)^2} =$
$= 8 - 2\sqrt{7} + \sqrt{7} + \sqrt{5} + \sqrt{7} - \sqrt{5} = 8.$

7. A mind-reader computer programme:

Play this game a number of times:
http://www.cut-the-knot.org/Curriculum/Magic/MindReaderNine.shtml
Can you explain what's going on?

Hint: Since we can't seriously believe that the computer is a mind reader, it must be that the programme assigns the same shape to all the possible answers to the problem. Play again and check the numbers having the same shape as your solution. Notice any special property? Now try to prove it!

Solution: All possible answers to the problem are divisible by 9.
Let's suppose that the number is written AB, with A the tens digits and B the unit digit.
Then the number is equal to $10A + B$ and the number read backwards is $10B + A$.
(Here $10A$ means $10 \times A$, similarly $10B$).
The difference between the number and the number read backwards is

$$10A + B - (10B + A)$$
$$= 10A + B - 10B - A$$
$$= 9A - 9B = 9(A - B),$$

Which is always divisible by 9!