

# Pretty Primes

## Summary

This is a discussion about the power of prime factorization in solving questions about integers.

## Resources

- 1 Practice Question Sheet per student.
- 1 Class outline question sheet per student.

## Questions/Suggestions?

If you plan to use this material, or if you would write to send us feedback, please email [a.mustata@ucc.ie](mailto:a.mustata@ucc.ie)

## References:

- Fomin, S. Genkin, I. Itenberg “Mathematical Circles (Russian Experience)”
- Questions from the Irish Mathematical Olympiad

## Pretty Primes - Class Outline

### 1. Locker Games



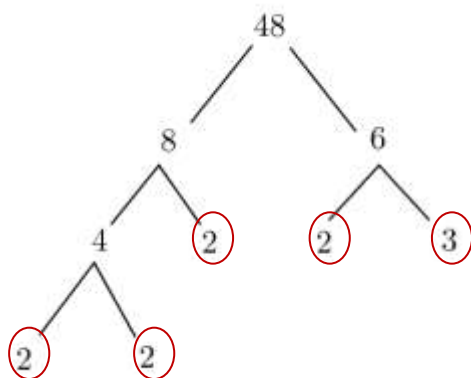
20 friends decided to decorate their school lockers. The first student puts a sticker on every locker. The second student puts a sticker on the second locker and then on every second locker after that. The third student then decorates the third locker and every third locker after that, and so on, until all 20 students are done decorating. They tally the number of stickers on each locker.

Locker	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Stickers																				
How many																				

- What does the number of stickers tell us about the corresponding locker number?
- Do you notice anything special about the lockers with 2 stickers? Explain.
- There are very few lockers with odd numbers of stickers. Do you notice anything special about them? Can you guess a general rule from here? Can you prove it?
- Look at the lockers 1, 2, 4, 8, 16. Anything special about their sticker numbers? Can you guess a general rule from here? Can you prove it?

### 2. Unique Prime Factorization

a)



This is referred to as a *factor tree* of 48.

The *prime factors* of 48 are the “fruits” at the end of the branches.

Hence  $48 = 2^4 \times 3$ .

This is called *prime factorising 48*.

Use this to calculate how many divisors 48 has.

- Find a general rule for calculating the number of divisors from the prime factorization of a number.
- Use the rule above to explain the following:  
*The square numbers are the only numbers with an odd number of divisors.*

### 3. Divisor rule revisited:

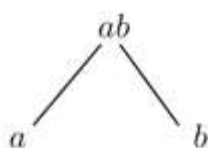
- a) Prove that the product of any three consecutive numbers is divisible by 6.
- b) Find all possible pair of digits  $x$  and  $y$  such that 36 divides the 4-digit number with digits  $x95y$ .

### 4. The Power of Products:

Find all possible values of a square number with the property that, if you add 15 to it, the result is still a square number.

### 5. An Important Property of Primes:

Suppose that we write a number as a product  $ab$ , and start a factor tree like this:



What does the Unique Prime Factorisation tell us about the prime factors of  $ab$ ?

(Think in terms of “hanging fruit”)



- a) Find all natural numbers  $n$  such that  $2n + 1$  is a multiple of 7.
- b) Your friend has chosen a five digit mystery number. She adds twice the number formed by the last two digits of the mystery number with three times the number formed by the first three digits (read from left to right). The result is divisible by 197. Prove that the mystery number is also divisible by 197.

For example, suppose the mystery number = 46295.  
 $2 \times 95 + 3 \times 462 = 1576 = 197 \times 8$ . You can also check that  $46295 = 197 \times 235$ .

- c) In the question b) above, is it important that  $n$  has 5 digits?

### 6. Checking for primes: a shortcut.

What's the fastest way to check whether 2011 is a prime?

### 7. How many primes?

- a) Prove that there are infinitely many primes “by contradiction”: Assume that there are only finitely many primes:  $p_1 = 2, p_2 = 3, p_3 = 5, \dots, p_n$  (unknown prime). What can we say about  $N = p_1 p_2 p_3 \dots p_n + 1$ ? (here  $p_1 p_2 \dots p_n = p_1 \times p_2 \times \dots \times p_n$ ).
- b) If  $p_1, p_2, p_3, \dots, p_n$  are the first  $n$  numbers, is  $N = p_1 p_2 p_3 \dots p_n + 1$  prime?

*A Tip for Maths Competitors:* Before you go into a maths competition, it may be a good idea to prime factorise the current year number. Especially if you're in a prime year, chances are you're going to use this in a problem 😊.

## Pretty Primes - Class Notes

### 1. Locker Games



This game can be played by the whole class even if you don't have lockers nearby. Partition the blackboard into 20 vertical regions and label them by consecutive numbers 1—20. These are your lockers.

20 friends decided to decorate their school lockers. The first student puts a sticker on every locker. The second student puts a sticker on the second locker and then on every second locker after that. The third student then decorates the third locker and every third locker after that, and so on, until all 20 students are done decorating. They tally the number of stickers on each locker.

- a) What does the number of stickers tell us about the corresponding locker number?
- b) Do you notice anything special about the lockers with 2 stickers? Explain.
- c) There are very few lockers with odd numbers of stickers. Do you notice anything special about them? Can you guess a general rule from here? Can you prove it?
- d) Look at the lockers 1, 2, 4, 8, 16. Anything special about their sticker numbers? Can you guess a general rule from here? Can you prove it?

*Solutions: a)*

Locker	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Stickers	1	2	2	3	2	4	2	4	3	4	2	6	2	4	4	5	2	6	2	6

b) *Hint:* Think about how each locker got its stickers.

*Answer:* There was one sticker added for each divisor (factor) of the locker number. The stickers count the number of divisors (factors) of the locker number.

*Notation:* We say that  $d$  is a divisor of  $a$  if  $a = d \times q$  for some integer number  $q$ . In this case we write  $d \mid a$  and say " $d$  divides  $a$ ."

c) The lockers with 2 stickers represent the numbers with exactly two divisors. These are known as the **prime numbers**. *Prime numbers* are numbers divisible *only* by 1 and themselves.

d) 1, 4, 9, 16 are all the square numbers between 1 and 20. We guess:

*The square numbers are the only numbers with an odd number of divisors.*

*Hint for proof:* If you can't prove the rule right away, first try it out on 36. What's a fast way to write all its divisors?

*Proof:* Write 36 as product of two numbers in all possible ways.

$$36 = 1 \times 36 = 2 \times 18 = 3 \times 12 = 4 \times 9 = 6 \times 6.$$

The factors are all the divisors of 36. They all come paired with a different factor except for 6 which is paired with itself exactly because 36 is the square of 6. The same argument works for all square numbers. By contrast, if a number is not a square number, each of its factors will come paired with a different factor from itself.

e)

Number	$1 = 2^0$	$2 = 2^1$	$4 = 2^2$	$8 = 2^3$	$16 = 2^4$
No. of divisors	1	2	3	4	5

Writing  $1 = 2^0$ ,  $2 = 2^1$ ,  $4 = 2^2$ ,  $8 = 2^3$ ,  $16 = 2^4$  we see that the sticker numbers are 1 unit larger than the powers (indices) of 2 in each locker number.

Why is that? Let's look at the divisors of 8:

$$1 = 2^0, \quad 2 = 2^1, \quad 4 = 2^2, \quad 8 = 2^3.$$

We can count them by their indices but we have to include 0. Hence, the number of divisors of  $2^n$  is  $n + 1$ .

The same pattern holds for  $3^n$ : it has  $n + 1$  divisors.

## 2. Unique Prime Factorization

In question e) it helped to write the numbers as powers of 2. This will not be possible for all numbers, but given one number, we can always write it as a product of smaller factors.

*Exercise:*

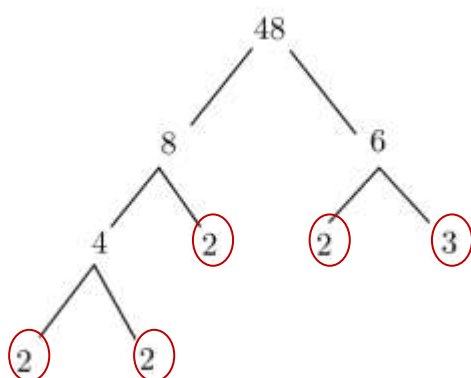
Write 48 as a product of (powers of) prime numbers: This is called prime factorising 48. Use this to calculate how many divisors 48 has.

*Solutions:*

a) Let's write our number as a product.

$$48 = 8 \times 6.$$

But 6 and 8 have their own divisors:  $8 = 2 \times 4$  and  $6 = 3 \times 2$ . And 4 has its own divisors:  $4 = 2 \times 2$ . We put the results up in a tree.



This is referred to as a *factor tree* of 48.

The *prime factors* of 48 are the "fruits" at the end of the branches.

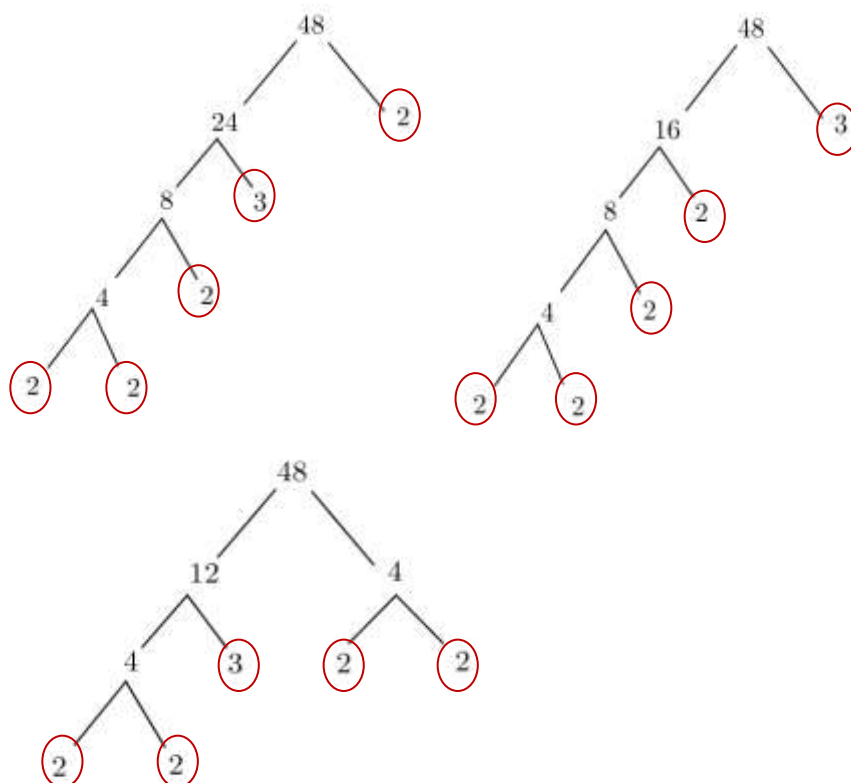
2 comes up 4 times and 3 once.

$$\text{All in all } 48 = 6 \times 8 = 3 \times 2 \times 4 \times 2 = 3 \times 2 \times 2 \times 2 \times 2 = 3 \times 2^4 = 2^4 \times 3.$$

(Note: no brackets are necessary because all operations are multiplications).

We can't factorise anymore, because 2 and 3 *are prime*. When we have expressed a number as a product of primes, the number is *prime factorised*.

But we can write 48 as a product in a number of ways. So different people can find different factor trees of 48. Let's check our various examples and compare the results:



There are other trees, but happily, they all have the same “hanging fruits”, so they all result in the same prime factorisation

$$48 = 3 \times 2^4 = 2^4 \times 3.$$

If we don't mind the order of the factors we can say that *48 has a unique prime factorization*.

We can write all divisors of 48 as product of primes and arrange them in a table according to the indices of 2 and 3:

$1 = 2^0 \times 3^0$	$2 = 2^1 \times 3^0$	$4 = 2^2 \times 3^0$	$8 = 2^3 \times 3^0$	$16 = 2^4 \times 3^0$
$3 = 2^0 \times 3^1$	$6 = 2^1 \times 3^1$	$12 = 2^2 \times 3^1$	$24 = 2^3 \times 3^1$	$48 = 2^4 \times 3^1$

There are 5 choices of indices for 2. Each of these can be paired with any of the 2 choices of indices of 3, so the result is a total of  $5 \times 2 = 10$  choices.

*b) Conclusions:*

**Unique Prime Factorisation:** If we don't mind the order of the factors, each *positive integer number can be written in a unique way as a product of powers of primes*:

$$a = 2^{k_2} \times 3^{k_3} \times 5^{k_5} \times 7^{k_7} \times 11^{k_{11}} \dots = 2^{k_2} 3^{k_3} 5^{k_5} 7^{k_7} 11^{k_{11}} \dots$$

Here  $k_p$  is just a notation for the index of  $p$ . Note that if  $k_p = 0$ , then  $p^{k_p} = p^0 = 1$ , so in this case  $p$  is not really a factor of  $a$ , and so we need not include it in the prime factorization of  $a$ :

$$48 = 2^4 \times 3^1 \times 5^0 \times 7^0 \times 11^0 \times \dots = 2^4 \times 3^1$$

**Divisor rule:** Given two numbers  $a = 2^{k_2} \times 3^{k_3} \times 5^{k_5} \times 7^{k_7} \times 11^{k_{11}} \dots$  and  $b = 2^{n_2} \times 3^{n_3} \times 5^{n_5} \times 7^{n_7} \times 11^{n_{11}} \dots$  then  $a$  divides  $b$  if and only if  $k_p \leq n_p$  for each prime  $p$ .

Example:  $48 = 2^4 \times 3$  divides  $b$  if  $2^4$  divides  $b$  and  $3$  divides  $b$ .

**Counting divisors:** Consider a number  $a$ , prime-factorised:

$$a = 2^{n_2} \times 3^{n_3} \times 5^{n_5} \times 7^{n_7} \times 11^{n_{11}} \dots$$

Then  $a$  has exactly  $(n_2 + 1)(n_3 + 1)(n_5 + 1)(n_7 + 1)(n_{11} + 1) \dots$  divisors.

Note that if  $n_p = 0$ , then  $(n_p + 1) = 1$  which doesn't really contribute anything to the product above.

c) All the indices have to be even. Indeed, if

$$x = 2^{n_2} \times 3^{n_3} \times 5^{n_5} \times 7^{n_7} \times 11^{n_{11}} \dots$$

then

$$x^2 = 2^{2n_2} \times 3^{2n_3} \times 5^{2n_5} \times 7^{2n_7} \times 11^{2n_{11}} \dots$$

Since every prime in the prime factorisation of  $x^2$  has an even index, when we add 1 we get an odd number. Multiplying all these odd numbers gives an odd number.

$$(2n_2 + 1)(2n_3 + 1)(2n_5 + 1)(2n_7 + 1)(2n_{11} + 1).$$

This is the number of divisors of  $x^2$ .

On the other hand, suppose that our number is not a square number. This means that one of the primes in its prime factorisation comes with an odd index. Add 1 and you get an even number. Multiply it by anything and the result is always even.

### 3. Divisor rule revisited:

A number  $a = 2^{n_2} \times 3^{n_3} \times 5^{n_5} \times 7^{n_7} \times 11^{n_{11}} \dots$  divides a number  $b$  if and only if:

$2^{n_2}$  divides  $b$ ,

$3^{n_3}$  divides  $b$ ,

$5^{n_5}$  divides  $b$  ...

*Exercise:*

a) Prove that the product of any three consecutive numbers is divisible by 6.

b) Find all possible pair of digits  $x$  and  $y$  such that 36 divides the 4-digit number with digits  $x95y$ .

*Solution:*

a) The prime factorisation of 6 is  $6 = 2 \times 3$ . By the Divisor rule, A number is divisible by 6 if it is divisible by 2 and by 3.

Among any three consecutive numbers there is at least one even number, and at least one number which is a multiple of 3. So their product will be divisible by both 2 and 3. The End.

b) The prime factorisation of 36 is  $36 = 2^2 \times 3^2$ . By the Divisor rule, A number is divisible by 36 iff it is divisible by  $2^2 = 4$  and by  $3^2 = 9$ .

The Divisibility by 4 rule implies that  $4 \mid x95y$  iff  $4 \mid (50 + y)$ .  
As  $50 + y = 48 + 2 + y$ , we get the cases: i)  $y = 2$ ; and ii)  $y = 6$ .

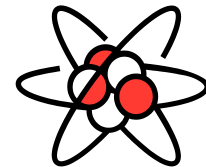
The Divisibility by 9 rule implies that  $9 \mid x95y$  iff  $9 \mid (x + 9 + 5 + y)$ .

Case i)  $y = 2$  and  $x + 9 + 5 + 2 = x + 16$  so  $x = 2$ .

ii)  $y = 6$  and  $x + 9 + 5 + 6 = x + 20$  so  $x = 7$ .

#### 4. The Power of Products:

Physicists and Chemists always wanted to know what stuff is made of: they found molecules, atoms, electrons, protons, neutrons, quarks, kept together by forces...



For numbers, we found:

*Every natural number is constructed out of its own set of primes with indices, "glued together" by multiplication.*

This rule is so powerful, that most of the problems where you have to find some natural numbers satisfying some special conditions are solved based on products and their close relatives: divisibility and prime factorisation.

That's because products are the key to the very structure of natural numbers.

**Example:**

Find all possible values of a square number with the property that, if you add 15 to it, the result is still a square number.

**Solution:**

Let's give the unknown number a name. Since it's a square number, we can call it  $x^2$ . When we add 15 to it we obtain another square number, let's call it  $y^2$ . So

$$y^2 = x^2 + 15.$$

It's impossible to find all solutions of this equation by trial and error:  $x$  could be a very big number, and then so could  $y$ .

What does this equation have to do with products?

We could factorise  $15 = 3 \times 5$ . Then let's get 15 all on its own on the right hand side of the equation:  $y^2 = x^2 + 15$  becomes  $y^2 - x^2 = 15$ .

Do you notice/remember anything special about  $y^2 - x^2$ ?

In the Distributivity class we discussed a nice trick based on the formula

$$y^2 - x^2 = (y - x)(y + x).$$

So our equation now becomes  $(y - x)(y + x) = 15 = 3 \times 5$ .



At this point we could identify  
the smaller factors on each side as  $y - x = 3$ ,  
and the larger factors as  $y + x = 5$ .

Add up the terms to get  $2y = 8$  so  $y = 4$ , and then  $x = 5 - 4 = 1$ .

So our solution is, not surprisingly,  $x^2 = 1$ .

To check our answer, we see that  $1+15=16$  is indeed a square number.

But is this the only solution?

We should remember that  $15 = 1 \times 15$  as well so another option is

$$y - x = 1,$$

$$y + x = 15.$$

Add up the terms to get  $2y = 16$  so  $y = 8$ , and then  $x = 15 - 8 = 7$ .

Check:  $49+15=64$ , a square number.

*Problem solvers beware:*

Some quick problem solvers may have found the solutions 1 and 49 with a few trials.

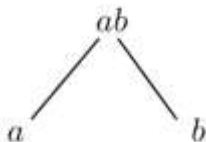
Will they get the big prize for their solution?

Answer: No, not from mathematicians for sure. When we solve a problem, we want to be absolutely sure that we have found all solutions. Think about clearing two land mines from a minefield... wouldn't you like to know that there are absolutely no other land mines hidden around?

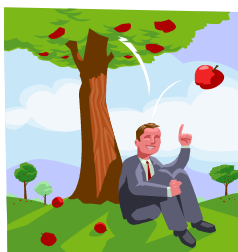
This last step is worth all the money in mathematics, because in this case you're eliminating infinitely many hidden possibilities – pretty hard job if you ask me!

## 5. An Important Property of Primes:

Suppose that we write a number as a product  $ab$ , and start a factor tree like this:



What does the Unique Prime Factorisation tell us about the prime factors of  $ab$ ? (Hint: think in terms of “hanging fruit”)



We can reformulate the Unique Prime Factorisation rule like this:

*If  $p$  is a fruit of  $ab$ , then it hangs from one of its branches  $a$  or  $b$ !*

More Mathematically put:

*Important Property of Primes!*

*Let  $p$  be a prime. If  $p \mid ab$ , then  $p \mid a$  or  $p \mid b$ .*

*Exercise:*

a) Find all natural numbers  $n$  such that  $2n + 1$  is a multiple of 7.

*Solution:*

A quick check will yield  $n = 3, 10, 17, 24, \dots$

We start to guess a pattern: we start from 3 and go every 7<sup>th</sup> number after that.

We can write such numbers as  $n = 3 + 7k$ , where  $k$  can be any natural number.

But how can we check/prove that these are all the solutions?

One natural first step would be to subtract 3 from our number  $n$ . This will mean subtracting 6 from  $2n$ , so

$$2n + 1 = 2n - 6 + 7 = 2(n - 3) + 7.$$

Now 7 already divides 7, so 7 divides  $2n + 1$  if and only if 7 divides  $2(n - 3)$ .

By the property of primes, if 7 divides  $2(n - 3)$  then 7 divides 2 or  $n - 3$ . Clearly 7 doesn't divide 2, so it must divide  $n - 3$ . In other words,  $n - 3 = 7k$ , where  $k$  can be any natural number.

*Exercise:*

b) Your friend has chosen a five digit mystery number. She adds twice the number formed by the last two digits of the mystery number with three times the number formed by the first three digits (read from left to right). The result is divisible by 197. Prove that the mystery number is also divisible by 197.

For example, suppose,  $n = \boxed{462}\boxed{95}$ .

$2 \times 95 + 3 \times 462 = 1576 = 197 \times 8$ . You can also check that  $46295 = 197 \times 235$ .

c) In the question b) above, is it important that  $n$  has 5 digits? Can you formulate a similar problem that works for numbers  $n$  with any number of digits?

*Solution:*

Let's denote the number formed by the last two digits by  $x$  and the number formed by the first three digits by  $y$ . How can we write the mystery number  $n$  as a formula in  $x$  and  $y$ ?

For example,  $46295 = 46200 + 95 = 95 + 100 \times 462$ .

In general,  $n = x + 100y$ .

Now the operations described in the problem are  $2x + 3y$ .

We know that 197 divides  $2x + 3y$  and we'd like to prove that 197 divides  $x + 100y$ .

Since in the first number,  $x$  comes up twice, it's natural to look at

$2n = 2(x + 100y) = 2x + 200y$ . We compare to  $2x + 3y$ :

$2n = 2x + 200y = 2x + 3y + 197y$ . Now we know that  $197 \mid (2x + 3y)$  and  $197 \mid 197y$ , so  $197 \mid (2x + 200y) = 2n$ . Since 197 doesn't divide 2, it must be that 197 divides  $n$ .

## 6. Checking for primes: a shortcut.

We saw before that it's important to be able to check if a number is prime as quickly as possible.

*Example:*

a) Is 2011 a prime?

To find factors of 2011 in pairs, we saw before that it's enough to check all the numbers smaller than  $45 > \sqrt{2011}$ . But is it important to check all these numbers, or only some of them?

Answers:

a) It's enough to check if any of the primes less than 45 are factors of 2011. Indeed, if 2011 has some factors other than 1 and itself, then it automatically has some prime factors other than 1 and itself!

You need to check if these numbers divide 2011:

[2](#) [3](#) [5](#) [7](#) [11](#) [13](#) [17](#) [19](#) [23](#) [29](#) [31](#) [37](#) [41](#) [43](#)

*A Tip for Maths Competitors:*

Before you go into a maths competition, it may be a good idea to prime factorise the current year number. If you're in a prime year, chances are you're going to use this in a problem 😊.

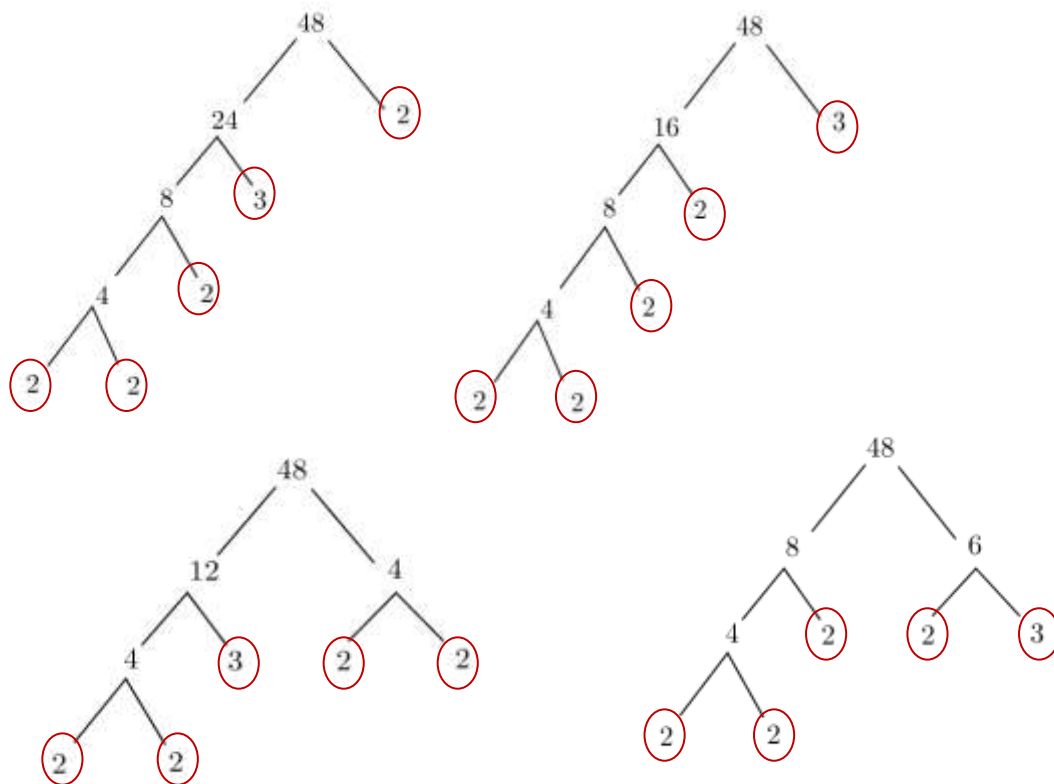
## 7. The Return of the Unique Prime Factorisation, or Panicky Primes

We return to the rule we discovered:

**The Unique Prime Factorisation:** If we don't mind the order of the factors, Any positive natural number can be written in a unique way as a product of primes with indices.

*Class survey/discussion:*

a) Think about the time when we drew different factor trees for 48 and we always found the same “hanging fruits” with same indices. Did this surprise you? If yes, why? If not, why not?



b) Think about the **Unique Prime Factorisation** rule. Are you convinced that it's true for all natural numbers? If yes, why? If not, why not?

c) As the name suggests, the **Unique Prime Factorisation** is due to the very special nature of prime numbers. *Let  $p$  be a prime. If  $p \mid ab$ , then  $p \mid a$  or  $p \mid b$ .*

Are you convinced that it's true for all primes? If yes, why? If not, why not?

*Moderator's Comments:*

a) Some people may not have been surprised by this discovery. This may be because they already knew all the factors of 48 and could quickly check that they have no prime factors other than 2 and 3. Or they may have noticed that 48 could not possibly have any prime factors other than 2 and 3: indeed, the only prime they needed to check would be 5, since 7 already satisfies

$$7 \times 7 = 49 > 48.$$

These checks can go on so quickly in our minds that we may find the results absolutely natural and obvious...

b) Even though a rule holds for 48 and other numbers small enough to check, this does not guarantee that the rule will be true for all numbers.

Suppose that we choose a really, really large number, so that our lives won't suffice to find **all** its factors.

Suppose that we manage to find a factor tree for that number.

(This is an easier task, since it's enough to find two factors, and then two factors of the factors, etc. The numbers decrease with each step.)

What guarantee do we have that there are no primes outside the tree, but which divide the number – like some fruits not hanging from the branches?

(They might be part of another factor tree of the same number!)

## Pretty Primes – Practice Questions

1. A class of 50 has been playing the locker game.
  - a) How many lockers have exactly 2 stickers each?
  - b) How many lockers have an odd number of stickers each?
  - c) Which lockers have exactly 3 stickers each?
  - d) Find a sequence of 6 lockers with consecutive numbers of stickers.
2. How many divisors do the following numbers have?
  - a)  $i) 108$                        $ii) 1000$                        $iii) 3,000,000$
  - b)  $i) 6^2 \times 6^3 \times 6^{11}$                $ii) 6^{10} \times 10^{15} \times 15^6$                $iii) 10 \times 9 \times 8 \times \dots \times 2 \times 1$
3. One day, every student in the galaxy gets bored and decides to play the locker game. The  $10^{10}$  students collect  $10^{10}$  lockers for the game.
  - a) Which lockers have exactly 7 stickers each?
  - b) Among the first 100 lockers, how many have exactly 12 stickers each?
  - c) Find a rule about the number of stickers on lockers which are cube numbers.Hint: It may involve the remainder when dividing this number by 3.
4. a) Find the smallest positive natural number whose product with 2016 is a square number.
  - b) Find the natural numbers  $x, y, z$  and  $t$ , all larger than 1, knowing that
$$3^x + 3^y + 3^z + 3^t = 6840.$$
  - c) Find the smallest mystery natural number  $m$  with the property that  $3m$  is the cube of a natural number and  $5m$  is the 5<sup>th</sup> power of a natural number.
5. a) For any natural number  $n$ , prove that  $n(n+1)(2n+1)$  is divisible by 6.
  - b) Prove that the product of any 5 consecutive numbers is divisible by 120.
6. a) Find all possible digits  $a$  and  $b$  for which the 4-digit number with digits  $2a9b$  is divisible by 12.
  - b) Find all possible digits  $a$  and  $b$  for which  $2^a \times 9^b$  is equal to the 4-digit number with digits  $2a9b$ .
7. a) Find all possible values of a square number with the property that, if you add 12 to it, the result is still a square number.
  - b) Find all natural numbers  $x$  for which  $1 + 99 \times 2^x$  is the square of a natural number.
  - c) Find all natural numbers  $x$  for which  $x^2 + 2x + 10$  is the square of a natural number.
8. Find all natural numbers with the property that adding 2014 to their cube gives another cube number.
9. Check if any of these numbers are prime:
  - $i) 2011$      $ii) 2013$      $iii) 2017$      $v) 2019$ .
10. Find all natural numbers  $n$  such that  $3n + 1$  is a multiple of 5.

**11.** A 5 digit mystery number has the property that the sum of its first, third and fifth digit is 19, while the sum of the second and fourth digit is 8. Prove that the number has at least 24 divisors.

*Hint:* Check divisibility by 9 and by 11 first. Can you prove that the number has 12 divisors? Where could the other 12 come from?

**12.** A 7 digit number is called *unlucky* if twice the number formed by its last two digits plus five times the number formed by its first five digits is a multiple of 13. Prove that an *unlucky* number is also divisible by 13.

**13.** (IrMO 2014) Each of the four positive integers  $N$ ;  $N + 1$ ;  $N + 2$ ;  $N + 3$  has exactly 6 positive divisors. There are exactly 20 different positive numbers which are exact divisors of at least one of the numbers. One of these is 27. Find all possible values of  $N$ . (Both 1 and  $m$  are counted as divisors of the number  $m$ .)

**14.** Show that there are 1000 consecutive numbers all of which are composite (that is, not prime).

**15.** a) Show that there are infinitely many prime numbers of the form  $3k + 2$ , where  $k$  is an integer.

b) Same for  $4k + 3$ , where  $k$  is integer.

c) Same for  $6k + 5$ , where  $k$  is integer.

**16.** Can one find 14 consecutive positive integers each of which is divisible by at least one of 2, 3, 5, 7, 11?

## Pretty Primes - Activity Sheet Solutions

1. a) 15 prime numbers are smaller than 50.

2   3   5   7   11   13   17   19   23   29   31   37   41   43   47

b) There are 6 square numbers smaller than 50, namely 4, 9, 16, 25, 36, 49.

c) 4, 9, 25, 49: the squares of primes have exactly 3 divisors.

d) 1, 2, 4, 8, 16, 32 or, more interestingly, 1, 7, 25, 27, 16

Prove that any number of the form  $n(n+1)(2n+1)$  is divisible by 6.

Solution:

One of the numbers  $n$ ,  $(n+1)$  is even. It remains to prove that among  $n$ ,  $(n+1)$ ,  $(2n+1)$  there is a number which is a multiple of 3. Let's see what happens if neither  $n$  nor  $n+1$  is a multiple of 3. Then  $n+2$  must be. So  $n+2 = 3k$  for some natural number  $k$ , or equivalently,  $n = 3k - 2$ .

Let's see what this says about  $(2n+1)$ :

$$2n+1 = 2(3k-2)+1 = 6k-4+1 = 6k-3 = 3(2k-1).$$

d) The number of divisors of a cube number always yields remainder 1 when divided by 3, because it is of the form

$$(3n_2+1)(3n_3+1)(3n_5+1)(3n_7+1)(3n_{11}+1).$$

e)  $2016 = 2^5 \times 3^2 \times 7^1$ . When we multiply 2016 by another number, we can either increase the indices of the primes 2, 3, 7, or bring in new primes. As we're looking for the smallest product, it makes sense to only modify indices so that they all become even: 5 becomes 6; 2 stays 2 and 1 becomes 2. For this, we multiply 2016 by  $2 \times 7 = 14$  and we get

$$2016 \times 14 = 2^5 \times 3^2 \times 7^1 \times 2 \times 7 = 2^6 \times 3^2 \times 7^2 = (2^3 \times 3 \times 7)^2.$$