

Pretty Primes – Practice Questions

1. A class of 50 has been playing the locker game.
 - a) How many lockers have exactly 2 stickers each?
 - b) How many lockers have an odd number of stickers each?
 - c) Which lockers have exactly 3 stickers each?
 - d) Find a sequence of 6 lockers with consecutive numbers of stickers.
2. How many divisors do the following numbers have?
 - a) $i) 108$ $ii) 1000$ $iii) 3,000,000$
 - b) $i) 6^2 \times 6^3 \times 6^{11}$ $ii) 6^{10} \times 10^{15} \times 15^6$ $iii) 10 \times 9 \times 8 \times \dots \times 2 \times 1$
3. One day, every student in the galaxy gets bored and decides to play the locker game. The 10^{10} students collect 10^{10} lockers for the game.
 - a) Which lockers have exactly 7 stickers each?
 - b) Among the first 100 lockers, how many have exactly 12 stickers each?
 - c) Find a rule about the number of stickers on lockers which are cube numbers.

Hint: It may involve the remainder when dividing this number by 3.
4.
 - a) Find the smallest positive natural number whose product with 2016 is a square number.
 - b) Find the natural numbers x, y, z and t , all larger than 1, knowing that
$$3^x + 3^y + 3^z + 3^t = 6840.$$
 - c) Find the smallest mystery natural number m with the property that $3m$ is the cube of a natural number and $5m$ is the 5th power of a natural number.
5.
 - a) For any natural number n , prove that $n(n+1)(2n+1)$ is divisible by 6.
 - b) Prove that the product of any 5 consecutive numbers is divisible by 120.
6.
 - a) Find all possible digits a and b for which the 4-digit number with digits $2a9b$ is divisible by 12.
 - b) Find all possible digits a and b for which $2^a \times 9^b$ is equal to the 4-digit number with digits $2a9b$.
7.
 - a) Find all possible values of a square number with the property that, if you add 12 to it, the result is still a square number.
 - b) Find all natural numbers x for which $1 + 99 \times 2^x$ is the square of a natural number.
 - c) Find all natural numbers x for which $x^2 + 2x + 10$ is the square of a natural number.
8. Find all natural numbers with the property that adding 2014 to their cube gives another cube number.
9. Check if any of these numbers are prime:
 $i) 2011$ $ii) 2013$ $iii) 2017$ $v) 2019$.
10. Find all natural numbers n such that $3n + 1$ is a multiple of 5.

11. A 5 digit mystery number has the property that the sum of its first, third and fifth digit is 19, while the sum of the second and fourth digit is 8. Prove that the number has at least 24 divisors. (positive and negative)

Hint: Check divisibility by 9 and by 11 first.

12. A 7 digit number is called *unlucky* if twice the number formed by its last two digits plus five times the number formed by its first five digits is a multiple of 13.

Prove that an *unlucky* number is also divisible by 13.

13. (IrMO 2014) Each of the four positive integers N ; $N + 1$; $N + 2$; $N + 3$ has exactly 6 positive divisors. There are exactly 20 different positive numbers which are exact divisors of at least one of the numbers. One of these is 27. Find all possible values of N . (Both 1 and m are counted as divisors of the number m .)

14. Show that there are 1000 consecutive numbers all of which are composite (that is, not prime).

15. a) Show that there are infinitely many prime numbers of the form $3k + 2$, where k is an integer.

b) Same for $4k + 3$, where k is integer.

c) Same for $6k + 5$, where k is integer.

16. Can one find 14 consecutive positive integers each of which is divisible by at least one of 2, 3, 5, 7, 11?

Pretty Primes - Class Notes

1. Locker Games



This game can be played by the whole class even if you don't have lockers nearby. Partition the blackboard into 20 vertical regions and label them by consecutive numbers 1—20. These are your lockers.

One day Arya brings a big roll of stickers to school and she and her classmates decide to play a game with their lockers. The first student puts a sticker on every locker. The second student puts a sticker on the second locker and then on every second locker after that. The third student then decorates the third locker and every third locker after that, and so on, until all 20 students are done decorating. They tally the number of stickers on each locker and start noticing all kinds of interesting things.

- Do the tally yourselves and write the Locker and Sticker numbers in a table. Do you notice any interesting patterns?
- What does the number of stickers tell us about the corresponding locker number?
- Do you notice anything special about the lockers with 2 stickers? Explain.
- There are very few odd numbers of stickers. Do you notice anything special about their lockers? Can you guess a general rule from here? Can you explain why the rule should be true?
- Look at the lockers 1, 2, 4, 8, 16. Anything special about their sticker numbers? Can you guess a general rule from here? Can you prove it?
- Do you notice any other patterns? Try various ideas.

Locker	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Stickers																				

2. Counting divisors in pairs.

Next Arya and her classmates thought how cool it would be if they had 3,529,147 lockers in their school ... think of how many stickers they would get to use on the last one! ... In truth, they had no idea how many.

Goal:

Find a systematic way to calculate the number of divisors of any number. Let's start small. Take 48 as an example.

Exercise: Method I for counting divisors:

a) Find all pairs of numbers whose product is 48.

For this we sift through numbers in increasing order, starting from 1. Once we find a divisor of 48, we write 48 as a product between that divisor and a second number. The second factor will be a divisor of 48 as well. Through how many numbers do we need to sift before we're sure we've exhausted all possibilities?

b) Replace 48 by an unknown number n . Through how many numbers do we need to sift before we know we've exhausted all ways of writing n as a product? Write your answer in terms of n .

Conclusions:

To find all the divisors of 3,529,147 by this method, we'd have to sift through $\sqrt{3,529,147}$ numbers. That's not too far from $\sqrt{4,000,000} = 2,000$ checks. A bit much!

3. Unique Prime Factorization

The first method can be very slow. To quicken it, let's put our engineer caps on: we'll first break our number into smaller parts and then construct our divisors out of the available components:

Exercise: Method II for counting divisors:

a) Write 48 as a product of (possibly more than two) prime numbers: This is called prime factorising 48. In how many ways can you do it?

b) Let d be any random divisor of 48.

i) What prime factors can d have?

ii) How often can each prime come up as factor when you prime factorise d ?

c) Use this to calculate how many divisors 48 has.

Conclusions:

a) **Unique Prime Factorisation:** If we don't mind the order of the factors, Any positive natural number can be written in a unique way as a product of primes with indices.

b) **Divisor rule:** Given two numbers a and b ,

a divides b if and only if the primes of a are among the primes of b , and their indices are at most equal to the indices in the prime factorisation of b .

c) **Counting divisors:** Consider a number a , prime-factorised.

To find how many divisors a has, add 1 to each index in the prime-factorisation, then multiply the results.

4. Counting divisors, Prime Factorisation: Applications

Exercise:

a) Remember the rule we found in the Locker Games:

The square numbers are the only numbers with an odd number of divisors.

Let's look at the prime factorisation of a square number:

- i) What can you say about the indices of all primes in the prime factorisation of a square number?
- ii) Can you use your observations on indices to write a new proof of the rule above?
- b) Can you find a rule about the number of divisors of a cube number?

Hint: It may involve the remainder when dividing this number by 3.

- c) Find the smallest positive natural number whose product with 2016 is a square number.

5. Divisor rule: Applications

We can also use the

Divisor rule: Given two numbers a and b ,

a divides b if and only if the primes of a are among the primes of b ,

and their indices are at most equal to the indices in the prime factorisation of b .

to find out if a number b is divisible by a given number a .

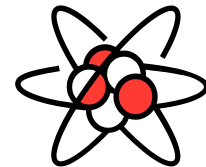
In this case we know the prime factorisation of the first number a . It remains to check if the second number b has those same prime factors, with at least the indices they have in a .

Exercises:

- a) i) Prove that the product of any three consecutive numbers is divisible by 6.
- ii) For any natural number n , prove that $n(n + 1)(2n + 1)$ is divisible by 6.
- b) Find all possible pair of digits x and y such that 36 divides the 4-digit number with digits $x95y$.

6. The Power of Products: Applications

Physicists and Chemists always wanted to know what stuff is made of: they found molecules, atoms, electrons, bosons and fermions, kept together by gravity, electromagnetic, strong or weak forces...



If we wanted to know what numbers are made of:

Every natural number is constructed out of its own set of primes with indices, "glued together" by multiplication.

This rule is so powerful, that many of the problems where you have to find some natural numbers satisfying some special conditions are solved based on products and their close relatives: divisibility and prime factorisation.

That's because products are the key to the very structure of natural numbers.

Exercise:

Find all possible values of a square number with the property that, if you add 15 to it, the result is still a square number.

Problem solvers beware:

Some quick problem solvers may have found the solutions 1 and 49 with a few trials. Will they get the big prize for their solution?

Answer: No, not from mathematicians for sure. When we solve a problem, we want to be absolutely sure that we have found all solutions. Think about clearing two land mines from a minefield... wouldn't you like to know that there are absolutely no other land mines hidden around?

This last step is worth all the money in mathematics, because in this case you're eliminating infinitely many hidden possibilities – pretty hard job if you ask me!

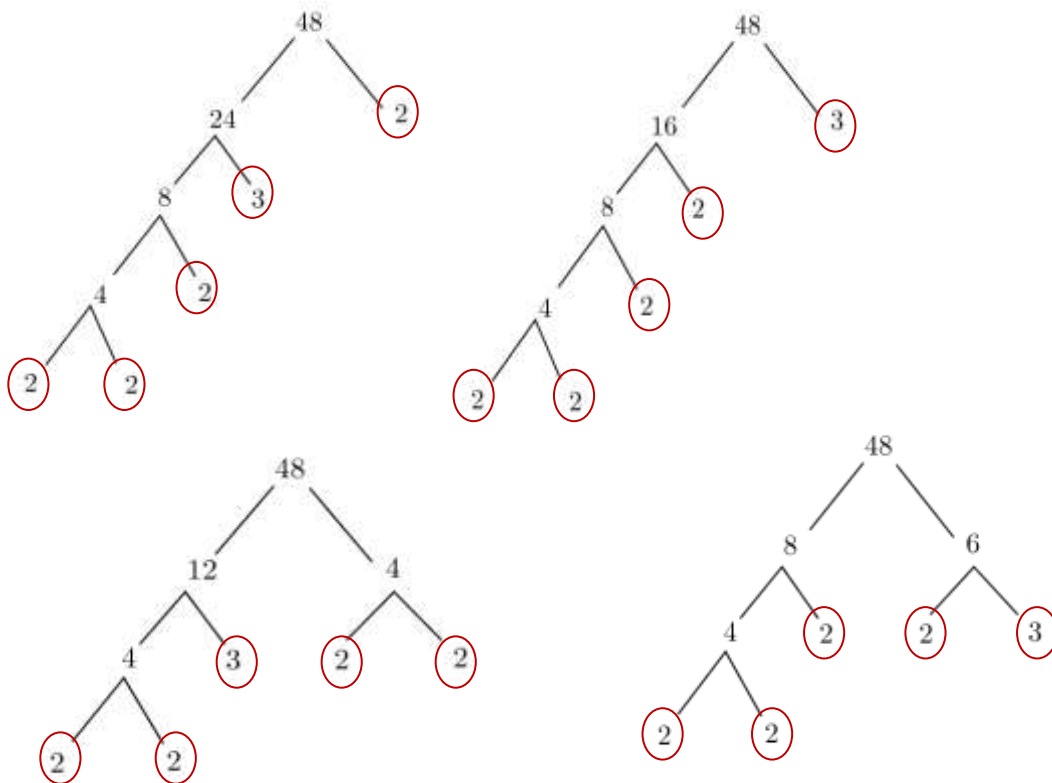
7. The Return of the Unique Prime Factorisation, or Panicky Primes

We return to the rule we discovered:

The Unique Prime Factorisation: If we don't mind the order of the factors, Any positive natural number can be written in a unique way as a product of primes with indices.

Class survey/discussion:

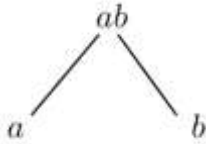
a) Think about the time when we drew different factor trees for 48 and we always found the same “hanging fruits” with same indices. Did this surprise you? If yes, why? If not, why not?



b) Think about the **Unique Prime Factorisation** rule. Are you convinced that it's true for all natural numbers? If yes, why? If not, why not?

c) As the name suggests, the **Unique Prime Factorisation** is due to the very special nature of prime numbers. Let's reformulate this rule in terms of primes:

Suppose that we write a number as a product ab , and start a factor tree like this:



What does the Unique Prime Factorisation tell us about the prime factors of ab ? (Hint: think in terms of “hanging fruit”)

Moderator's Comments:

a) Some people may not have been surprised by this discovery. This may be because they already knew all the factors of 48 and could quickly check that they have no prime factors other than 2 and 3. Or they may have noticed that 48 could not possibly have any prime factors other than 2 and 3: indeed, the only prime they needed to check would be 5, since 7 already satisfies

$$7 \times 7 = 49 > 48.$$

These checks can go on so quickly in our minds that we may find the results absolutely natural and obvious...

b) Even though a rule holds for 48 and other numbers small enough to check, this does not guarantee that the rule will be true for all numbers.



Suppose that we choose a really, really large number, so that our lives won't suffice to find **all** its factors.

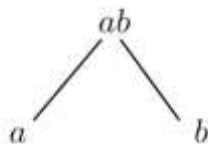
Suppose that we manage to find a factor tree for that number.

(This is an easier task, since it's enough to find two factors, and then two factors of the factors, etc. The numbers decrease with each step.)

What guarantee do we have that there are no primes outside the tree, but which divide the number – like some fruits not hanging from the branches?

(They might be part of another factor tree of the same number!)

c)



We can reformulate the Unique Prime Factorisation rule like this:

If p is a fruit of ab , then it hangs from one of its branches a or b !

More Mathematically put:

Important Property of Primes!

Let p be a prime. If $p \mid ab$, then $p \mid a$ or $p \mid b$.

Conclusions:

The following scary questions follow directly from our discussion:

d) Is Unique Prime Factorisation true even for very large numbers, with very large prime factors?

e) How large can prime numbers get anyway? Are there infinitely many primes?

f) We reformulated the Unique Prime Factorisation rule as a property of primes:

If $p \mid ab$, then $p \mid a$ or $p \mid b$.

Is this property true for all primes? Where does it come from? Can we prove it?

You can start panicking now!

Answers:

e) There are indeed infinitely many primes, and we'll prove this in a future lesson not too far from now.

f) Is also true, and it is due to way we divide natural numbers with remainder, as well as from the definition of primes. If you're really really curious, we will look at a proof in a future lesson, after we discuss the gcd (greatest common divisor, or highest common factor) of two numbers.

Because f) is true, then d) is true as well.

8. Checking for primes: a shortcut.

We saw before that it's important to be able to check if a number is prime as quickly as possible.

Examples:

a) Is 2011 a prime?

To find factors of 2011 in pairs, we saw before that it's enough to check all the numbers smaller than $45 > \sqrt{2011}$. But is it important to check all these numbers, or only some of them?

b) What are the prime factors of 3,529,147 ? (not really!)

A Tip for Maths Competitors:

Before you go into a maths competition, it may be a good idea to prime factorise the current year number. If you're in a prime year, chances are you're going to use this in a problem 😊.

9. An Important Property of Primes: Applications

Let p be a prime. If $p \mid ab$, then $p \mid a$ or $p \mid b$.

Exercise:

a) Find all natural numbers n such that $2n + 1$ is a multiple of 7.

b) Your friend has chosen a five digit mystery number. She adds twice the number formed by the last two digits of the mystery number with three times the number formed by the first three digits (read from left to right). The result is divisible by 197. Prove that the mystery number is also divisible by 197.

For example, suppose, $n = \boxed{462}\boxed{95}$.

$2 \times 95 + 3 \times 462 = 1576 = 197 \times 8$. You can also check that $46295 = 197 \times 235$.

c) In the question b) above, is it important that n has 5 digits? Can you formulate a similar problem that works for numbers n with any number of digits?

Pretty Primes - Activity Sheet

1. A class of 50 has been playing the locker game.
 - a) How many lockers have exactly 2 stickers each?
 - b) How many lockers have an odd number of stickers each?
 - c) Which lockers have exactly 3 stickers each?
 - d) Find a sequence of 6 lockers with consecutive numbers of stickers.
 - e) Find a sequence of 6 lockers with consecutive numbers of stickers under the condition that, with the exception of the first student, all other students placed a sticker on at most 1 of the 6 lockers.

2. Write these numbers as products of natural numbers in all possible ways:

a) 35, b) 36, c) 49, d) 54, e) 60.

3. How many divisors do the following numbers have?

a) i) 2^{10} ii) 11^8 iii) 17^{31}
 b) i) $2^4 \times 3^6$ ii) $5^5 \times 7^8$ iii) $11^{13} \times 13^{11}$ iv) 3×17^{17}
 c) i) 98 ii) 108 iii) 256 iv) 1000 v) 1,000,000
 d) i) $6^2 \times 6^3 \times 6^{11}$ ii) $6^{10} \times 10^{15} \times 15^6$ iii) $10 \times 9 \times 8 \times \dots \times 2 \times 1$

4. One day, every student in the galaxy gets bored and decides to play the locker game. The 10^{10} students collect 10^{10} lockers for the game.
 - a) How many stickers will there be on the last locker?
 - b) Among the first 100 lockers, which have exactly 7 stickers each?
 - c) Among the first 100 lockers, which have exactly 12 stickers each?

5.
 - a) Prove that the product of any 5 consecutive numbers is divisible by 120.
 - b) Find all possible digits a and b for which the 4-digit number with digits $2a9b$ is divisible by 12.
 - c) Find all possible digits a and b for which $2^a \times 9^b$ is equal to the 4-digit number with digits $2a9b$.
 - d) A 5 digit mystery number has the property that the sum of its first, third and fifth digit is 19, while the sum of the second and third digit is 8. Prove that the number has at least 24 divisors.
Hint: Check divisibility by 9 and by 11 first. Can you prove that the number has 12 divisors? Where could the other 12 come from?

6.
 - a) Find all possible values of a square number with the property that, if you add 12 to it, the result is still a square number.
 - b) "There exists no square number with the property that, if you add n to it, the result is still a square number."
 - i) Give 2 examples of natural numbers n for which this statement is true.
 - ii) Can you describe all natural numbers n for which this statement is true?
 - c) Find all natural numbers x for which $1 + 99 \times 2^x$ is the square of a natural number.

7. a) Prove the tongue-twister:
The divisor of a divisor of a number is also a divisor of the number.

b) Here is an example of three natural numbers such that the first divides the product of the second and third, but it divides neither the second, nor the third number: 6 divides (4×9) but 6 doesn't divide 4, and 6 doesn't divide 9.

Find other examples.

c) Recall the *Property of Primes*: Let p be a prime. If $p \mid ab$, then $p \mid a$ or $p \mid b$. Let p and q be two mystery prime numbers with the property that q^{10} is a multiple of p . Prove that q and p are equal.

8. a) Check if any of these numbers are prime:

i) 2011 ii) 2013 iii) 2015 iv) 2017 v) 2019.

9. a) Find all natural numbers n such that $3n + 1$ is a multiple of 5.

10. a) A 7 digit number is called *unlucky* if twice the number formed by its last two digits plus five times the number formed by its first five digits is a multiple of 13. Prove that an *unlucky* number is also divisible by 13.