Number Bases

Summary

This lesson is an exploration of number bases. There are plenty of resources for this activity on the internet, including interactive activities. Please feel free to supplement the material here with examples from those resources. The first activity sheet can be covered fairly quickly. The questions are meant to lead to the discovery of the expression

$$a_n \times 2^n + a_{n-1} \times 2^{n-1} + \ldots + a_2 \times 2^2 + a_1 \times 2 + a_0$$

and its consequences rather than mechanical calculations.

Ideally this should lead to discussions on polynomials – see Polynomials Question Sheet.

The games in the Games with Number Bases Sheet could be used at any time if students get bored of computational exercises.

Resources

- 1 Binary Bases and Explore Other Bases Question Sheet per student.
- 1 Number Bases and Polynomials Question Sheet per student.
- 1 Games with Number Bases Sheet per student.
- Scissors and Glue to go with the games sheet.
- Nim sticks or counters
- Solution Sheets for tutors
- Projector and Internet connection for the weblinks in the text – ready for quick access.

Questions/Suggestions?

If you plan to use this material, or if you would write to send us feedback, please email a.mustata@ucc.ie

Or write to:

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References:

-Fomin, S. Genkin, I. Itenberg "Mathematical Circles (Russian Experience)"
-Cut-the-Knot website
- Nrich website: http://nrich.maths.org/402
- Websites referenced in the text

Binary Basics

1. Imagine that you can only count with the digits 1 and 0, how would you be able to add, subtract or do anything else that you normally do with numbers whose digits go from 1 to 10?
A computer can only store information as 0-s or 1-s but it can store lots of these. E.g. the number 123 is stored as 1111011 on the computer. We call these **binary** numbers.

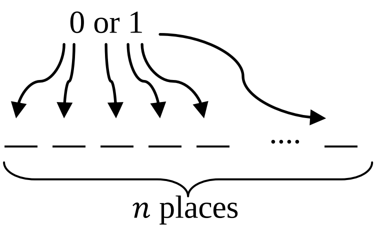
a) Look at the table to the right.
Can you figure out what the next numbers in the Binary number column are?

Decimal number	Binary number
0	0
1	1
2	10
3	11
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	
16	

b) Fill in the table and explain why the pattern holds.

Decimal number	Binary number
2	10
4	
8	
16	
32	
64	
128	
256	
2^n	

Hint: think of how many binary numbers can be made of at most n digits:



c) Look up these additions in the table above to the right.
Are you surprised? Can you explain what's going on?

Decimal	Binary
+ 1	1
2	10
3	

Decimal	Binary
+ 3	
4	
7	

Decimal	Binary
+8	
8	
16	

Decimal	Binary
	+1011
	101

Check out the binary counter here: http://www.mathsisfun.com/binary-number-system.html

How to Show that a Number is Binary: follow it with a little 2 like this: 101_2

d) Translate these binary numbers into decimal numbers:
$i)\ 1000001_2$ $ii)\ 1001001_2$ $iii)1010101_2$ $iv)\ 111111_2$

e) Write each of these as binary numbers:
$i)\ 100$ $ii)\ 200$ $iii)\ 800$ $iv)1000$ $v)\ 2013$

f) A number is written in binary like this: 1101100111_2. Without translating it into decimal notation, what is the remainder of the number when divided by: $i)\ 2?$ $ii)4?$ $iii)8?$ $iv)\ 64?$
(v) What is the quotient when the same number is divided by 64?

g) Can you explain why the base conversion method described here works:
http://www.purplemath.com/modules/numbbase.htm

2. A computer programmer at a party exclaims:

There aren't so many people here:
I could count us all on the fingers of one hand.

How many people are at the party?

3. Can you rewrite the sum $1 + 2 + 2^2 + 2^3 + \ldots + 2^{n-1}$ as a shorter formula?

4. Binary Guessing Game:

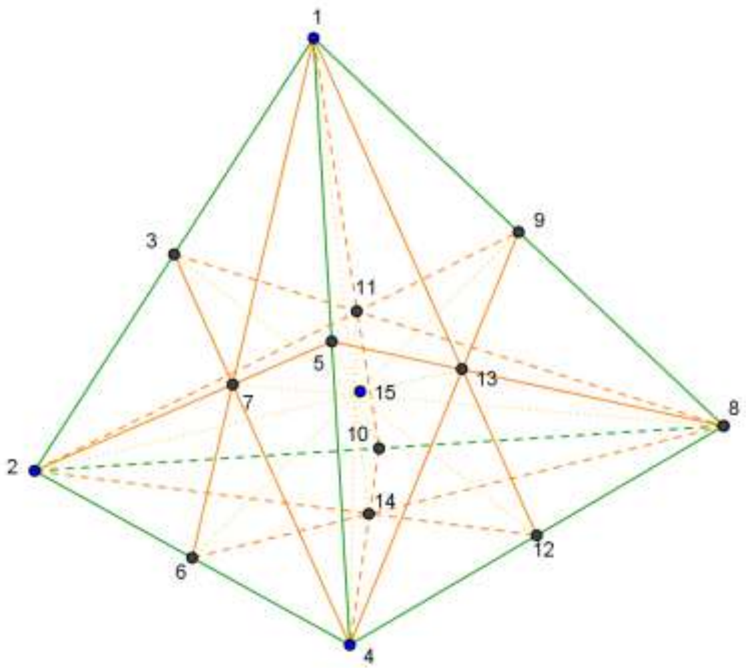
a) Build a binary tetrahedron using the net on the next page and look out for patterns:
i) on the vertices ii) on each edge iii) on the faces

b) For each vertex, we write down all the numbers connected to that vertex by one segment. We obtain the sets A, B, C, D below. Describe a defining rule for each of these sets:

A: 1; 3; 5; 7; 9; 11; 13; 15
B: 2; 3; 6; 7; 10; 11; 14; 15
C: 4; 5; 6; 7; 12; 13; 14; 15
D: 8; 9; 10; 11; 12; 13; 14; 15

If I pick a number between 1 and 15 and tell you exactly in which of the sets above it is, can you tell me the number and its binary form without looking at the tetrahedron? (you can try by looking first).

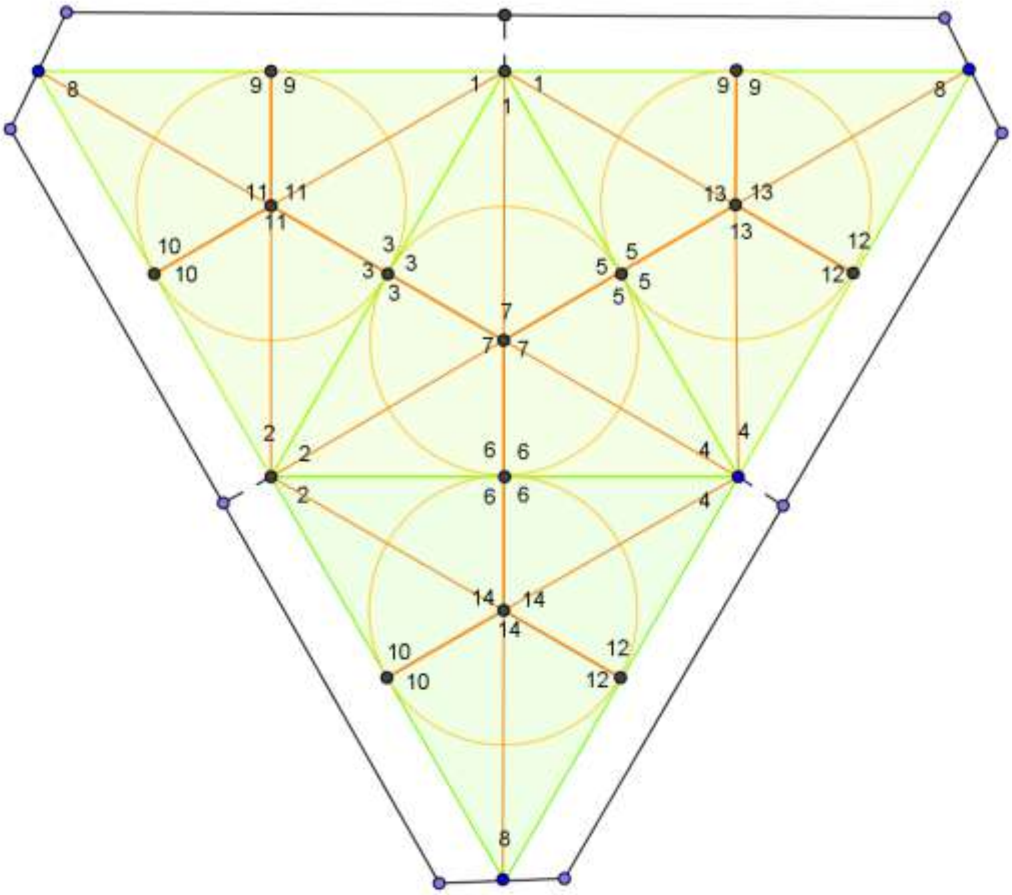
Keep this tetrahedron handy, we will use it when we play the game of Nim.



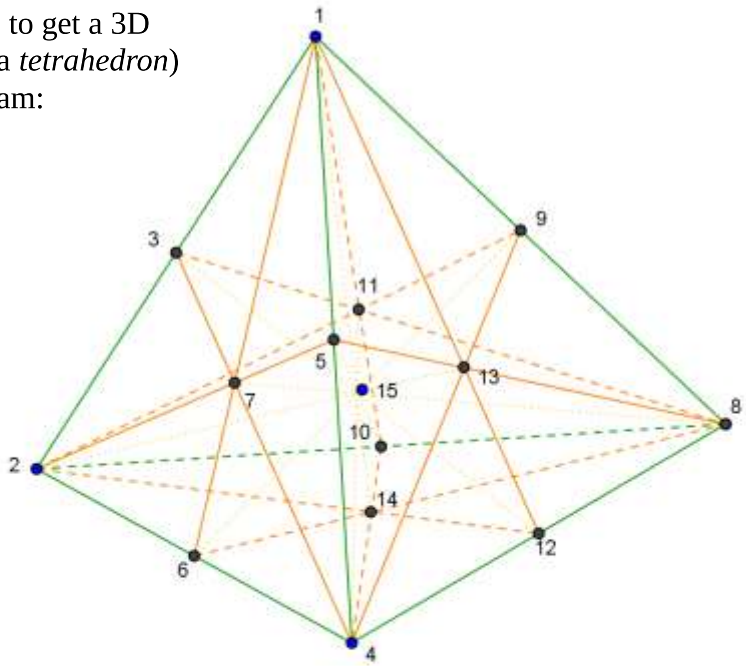
Note: the number 15 in the interior of the tetrahedron could not be included on the 2D net.

Binary Tetrahedron Construction:

Preparation: Using coloured pen, convert the numbers in this triangle into binary:



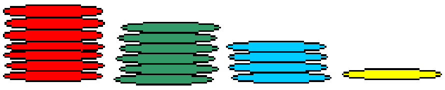
Cut and glue the borders to get a 3D triangular pyramid (aka a *tetrahedron*) like the one in this diagram:



Note: the number 15 in the interior of the tetrahedron could not be included on the 2D net.

2. Nim with piles

Nim is a game of strategy. There are many variants but we will try this one: Start with any number of counters in any number of piles. Two players take turns to remove any number of counters from a single pile. The winner is the player who takes the last counter.



For each of the following starting position, decide who wins if **both players play the best possible moves**. Once you convince yourself that the loser couldn't have played any better, place a V in the winner's column.

a) Playing with two piles:

Counters in each pile	1st player wins	2nd player wins
1, 1		
1, 2		
2, 2		
2, 5		
3, 3		

i) What are all the possible LOSE positions when playing with 2 piles?

ii) What are all the possible WIN positions when playing with 2 piles?
Describe a winning strategy.

b) Playing with three piles:

Counters in each pile	1st player wins	2nd player wins
1, 1, 3		
1, 2, 2		
1, 2, 3		
1, 3, 4		
1, 4, 5		

Starting with the smaller numbers and moving on to bigger ones, take any 3 numbers on the binary tetrahedron.
i) Who will win if the 3 numbers are all on the same line?
ii) Who will win if the 3 numbers include 2 vertices but are not all on the same line?
iii) Find more starting positions which insure that the 1st player loses.

c) Playing with four piles:

Counters in each pile	1st player wins	2nd player wins
1, 1, 2, 2		
1, 1, 2, 3		
m,m,n,n		

Using the binary tetrahedron above, can you find more 4 pile positions which insure that the 1st player will lose?
How about when the 1st player wins?

d) Look at all the positions discovered in the steps above in which the 2nd player wins. Write the numbers of counters in each pile in a column and convert them to binary. Do you notice any patterns?

Example: 1, 3, 5, 7 is one such position.

$$1 = 001_2$$
$$3 = 011_2$$
$$5 = 101_2$$
$$7 = 111_2$$

Other Number Bases

1. Here are a few:

Number Base	Uses symbols:	Counts in powers of:
Ternary	0, 1, 2	
Quaternary	0, 1, 2, 3	
Octal	0, 1, 2, 3, 4, 5, 6, 7	
Hexadecimal	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F	

2. Set the counter on the following webpage to different number bases. Stop the counter at random and write the number reached in decimal notation.
http://www.mathsisfun.com/binary-decimal-hexadecimal.html

3. Because 8 and 16 are powers of 2, they are also convenient ways to package information stored in computers.

a) Convert these binary numbers to the required bases:

(i) $0\,0\,1\,0\,1\,0\,0\,1\,1\,0\,0\,0\,1\,0\,0\,1\,1\,0\,1\,1\,1_2$ into octal.

(ii) $0\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,0\,1\,0\,0\,1\,1\,0\,1\,0\,1\,1\,0\,1\,1\,1\,1\,1_2$ into hexadecimal.

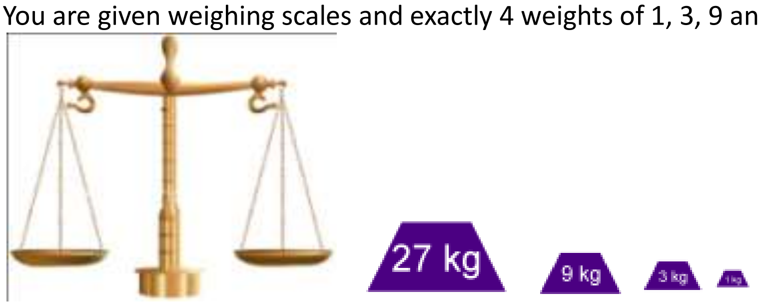
b) Why do programmers always mix up Halloween and Christmas?

c) Explore the use of hexadecimals in colour codes for your computer:
http://www.mathsisfun.com/hexadecimal-decimal-colors.html

d) What is $777777 + 1$ in octal? Translate the equation in decimal.

4. Weighing Game:

You are given weighing scales and exactly 4 weights of 1, 3, 9 and 27kg, like in the picture.



a) Using these, can you make the following measurements: (i) 2kg (ii) 6kg (iii) 18kg (iv) 24kg
b) What are all the possible weight measurements you can make with the above?
c) Repeat the problem with the weights of 1, 1, 5, 5, 25, 25 and 125kg.

5. Throughout history, various peoples have used various number systems. Find out about Babilonian numerals here: https://www.ncetm.org.uk/resources/13733

Number Bases and Polynomials

1. Mystery Basis:

An evil king wrote three secret two-digit numbers a; b; c. A handsome prince must name three numbers X; Y; Z, after which the king will tell him the sum $aX + bY + cZ$. The prince must then name all three of the King's numbers, or he will be executed. Help out the prince!

2. Best basis:

a) Translate this decimal number addition in a more suitable basis. Then translate the answer back to decimals.

$(i)\ 5(1+6+6^2+\ldots+6^{n-1})+1 \qquad (ii)\ 6\,(1+7+7^2+\ldots+7^{n-1})+1$

b) Can you write $X^n - 1$ as a product of two polynomials? Explain why this works.

3. Any basis?

Having been abducted by aliens from the exoplanet Xari, after an extraordinary journey I was confronted by an extremely angry court martial who accused my species of priding itself with excessive knowledge of prime numbers. There was only one way to redeem myself and that was to determine whether $\mathbf{2323}$ is a prime: a question they have been stumped with for too long. The question seemed simple enough, however I was terrified of getting it wrong, because I had no idea what number base they were using!

Still, after thinking for a minute, I confidently stated that 2323 is not prime, as $2323 = 23 \times 101$. They were happy with the answer so they let me go.

a) Was I just plain lucky, or does this work in any number basis using the symbols 0,1,2,3? Does it matter if they read their numbers left-to-right or right-to-left?

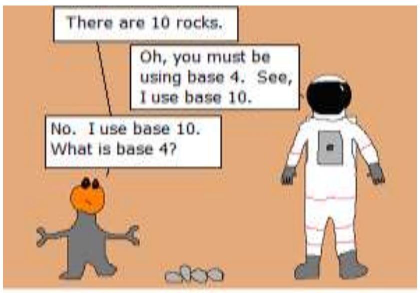
b) Are these numbers primes or composites on Xari:

$(i)1224?$ $\qquad (ii)1001?$ $\qquad (iii)\ 2332?$

I still have no idea of the number basis on Xari, but the numbers above exist there as such.

4. More Aliens

The Alien story was inspired by the following link.
Try the last puzzle there:
http://www.artofproblemsolving.com/Forum/blog.php?b=65



Every base is base 10.

5. Mystery polynomials

a) Find a polynomial with nonnegative integer coefficients such that $P(2) = 100$ and $P(1) = 3$. Prove that there is only one such polynomial.

b) Suppose $P(x)$ is an unknown polynomial, of unknown degree, with nonnegative integer coefficients. You have access to an oracle that, given an integer m, spits out $P(m)$, the value of the polynomial at m. However, the oracle charges a fee for each such computation, so you want to minimize the number of computations you ask the oracle to do. Show that it is possible to uniquely determine the polynomial after only two consultations of the oracle.

Binary Basics Solutions /Class discussion

1. Imagine that you can only count with the digits 1 and 0, how would you be able to add, subtract or do anything else that you normally do with numbers whose digits go from 1 to 10?
A computer can only store information as 0-s or 1-s but it can store lots of these. E.g. the number 123 is stored as 1111011 on the computer. We call these **binary** numbers.

How to Show that a Number is Binary

To show that a number is a *binary* number, follow it with a little 2 like this: **101_2**
This way people won't think it is the decimal number "101" (one hundred and one).

a) Look at the table to the right.

Can you figure out what the next numbers in the Binary number column are?

Hint: The rule you might use if forced to work with 0-s and 1-s only:

Informal Binary Counting Rule:
Start counting as usual. When you run out of options, then simply jump to the next smallest available number.

For example we counted 0, 1 as usual and then we ran out of digits, so we started our two digit binary numbers with 10, followed by 11 naturally. Then we ran out of options for two digit numbers, so we naturally started our three digit numbers with 100.

Solutions:

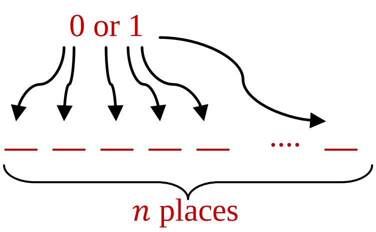
Decimal number	Binary number
0	0
1	1
2	10
3	11
4	100
5	101
6	110
7	111
8	1000
9	1001
10	1010
11	1011
12	1100
13	1101
14	1110
15	1111
16	10000

If this doesn't work for students, try:
http://www.mathsisfun.com/binary-number-system.html

b) Fill in the table and explain why the pattern holds.

Decimal number	Binary number
2	10
$4 = 2^2$	100
$8 = 2^3$	1000
$16 = 2^4$	10000
$32 = 2^5$	100000
$64 = 2^6$	1000000
$128 = 2^7$	10000000
$256 = 2^8$	100000000
$512 = 2^9$	1000000000
$1024 = 2^{10}$	10000000000
2^n	1 with n zeroes

Hint: think of how many binary numbers can be made of at most n digits:



Solution: There are $2 \times 2 \times \ldots \times 2 = 2^n$ binary numbers made of at most n digits, starting with 0 and ending with 1111...1. So the binary number 1111...1= $2^n - 1$ (because we started counting at 0, remember?) The next binary number, 1000...0 (n zeroes), in decimal notation is exactly 2^n.

c) Look up these additions in the table above to the right. Can you explain what's going on?

Decimal	Binary
+ 1	1
2	10
3	11

Decimal	Binary
+ 3	+11
4	100
7	111

Decimal	Binary
+8	+1000
8	1000
16	10000

Decimal	Binary
11	+1011
5	101
16	10000

We notice that the rules of addition are the same in both columns, including the "carry over" rule, with the difference that **in binary** 1+1=10 which one can get used to ☺.
The first two additions are not surprising, in view of the basic intuition of what it means to add and of the Informal Binary Counting Rule above.

Indeed the next binary number after 10_2 is obviously 11_2

Decimal	Binary
+ 1	1
2	10
3	11

and the 3rd binary number after 100_2 is 111_2,
because the 3rd binary number after 00_2 is 11_2,
(inserting an 1 as prefix does not alter the counting).

Decimal	Binary
+ 3	+11
4	100
7	111

This can be shown best in the long table from part a) above.
We checked that the binary $111_2 = 100_2 + 10_2 + 1_2$ corresponds to the decimal $7 = 4 + 2 + 1$.
We can generalize this argument to any binary number written with some digits a_n, a_{n-1}, $\ldots, a_2, a_1, a_0$. Each of these digits is either 0 or 1. The indices from 0 to n are just means to remember the placement of the digits in the number we wish to form. Thus in **binary**:

$$(a_n a_{n-1} \ldots a_2\, a_1\, a_0)_2 = \begin{array}{r} +\ (a_n\ 0\ \ 0\ldots\ 00)_2 \\ (a_{n-1}\,0\ \ldots\ 00)_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ (a_2 00)_2 \\ (a_1 0)_2 \\ (a_0)_2 \end{array}$$

Which in decimal notation is $a_n \times 2^n + a_{n-1} \times 2^{n-1} + \ldots + a_2 \times 2^2 + a_1 \times 2 + a_0$.
This explains why the addition rule (including carry over) works as

$$\begin{array}{r} +\ a_n \times 2^n + a_{n-1} \times 2^{n-1} + \ \ldots + a_2 \times 2^2 + a_1 \times 2 + a_0 \\ b_n \times 2^n + b_{n-1} \times 2^{n-1} + \ldots \quad + b_2 \times 2^2 + b_1 \times 2 + b_0 \end{array}$$

Are added column by column, and if $2^m + 2^m = 2^{m+1}$ is found in the $\_ \times 2\hat{}m$ column, it carries the $1 \times 2^{m+1}$ over into the next column to the left.
This may be abstract for some students, however it's good preparation for working with polynomials. Some other simple examples can be used to illustrate the abstract notations above.
To make it more practical we may hand it over to the binary counter found midpage here:
http://www.mathsisfun.com/binary-number-system.html
This website in addition has some interactive exercise.

d) Translate these binary numbers into decimal numbers :

$i)\ 1000001_2$ $\quad ii)\ 1001001_2$ $\quad iii) 1010101_2$ $\quad iv)\ 111111_2$

Solution: $i)\ 1000001_2 = 1000000_2 + 1_2 = 2^6 + 1 = 65.$

$ii)\ 1001001_2 = 2^6 + 2^3 + 1 = 73$
$iii) 1010101_2 = 2^6 + 2^4 + 2^2 + 1 = 85$
$iv)\ 111111_2 = 1000000_2 - 1_2 = 2^6 - 1 = 63.$

e) Write each of these as binary numbers:

$i)\ 100$ $ii)\ 200$ $iii)\ 800$ $iv) 1000$ $v)\ 2013$

Solution: $i)\ 100\ = 64 + 32 + 4 = 1100100_2$

Alternatively and perhaps after the fact,

$100 = 4 \times 25 = 4 \times (16 + 8 + 1) = 2^2 \times (2^4 + 2^3 + 1) = 2^6 + 2^5 + 2^2 = 1100100_2$,

Where we note that multiplication by 4 just added two zeroes at the end of the number $25 = 11001_2$.

By the same principle

$ii)\ 200 = 2 \times 100 = 11001000_2$
$iii)\ 800 = 2^3 \times 100 = 1100100000_2$
$iv)\ 1000 = +1100100000_2$
$\qquad\qquad 11001000_2$
$\qquad = 1111101000_2$

$v) 2013 = 2 \times 1000 + 13$ which with the help of the calculations and table above is:

$+\ 11111010000_2$
$\qquad\qquad 1101_2$
$= 11111011101_2$

f) A number is written in binary like this: 1101100111_2. Without translating it into decimal notation, what is the remainder of the number when divided by: $i)\ 2?$ $ii) 4?$ $iii) 8?$ $iv)\ 64?$
(v) What is the quotient when the same number is divided by 64?

Solution: i) 1 because $1101100111_2 = 1101100110_2 + 1_2$ and 1101100110_2 is a multiple of $10_2 = 2.$

$ii)$ The last two digit cut-off is $11_2 = 3$. That's because 1101100100_2 is a multiple of $100_2 = 4.$

$iii)$ The last three digit cut-off is $111_2 = 7.$

$iv)\ 64 = 2^6$. The last six digit cut-off is $100111_2 = 32 + 7 = 39.$

$v)$ We cut the remainder 100111_2 off the tail of 1101100111_2 and are left with $1101_2 = 13.$

Indeed, $1101100111_2 = 1101000000_2 + 100111_2 = 13 \times 64 + 39.$

Scroll down the page to get to the base conversion algorithm here:
http://www.purplemath.com/modules/numbbase.htm

At each step, the remainder is the last digit of the number when written in binary, and the quotient is obtained in binary by cutting the last digit off the given number. So the original number can be reassembled by reading the remainders downwards.

2. A computer programmer at a party exclaims:

There aren't so many people here:
I could count us all on the fingers of one hand.

How many people are at the party?

Solution: Well assuming an extended finger to mean 1, we get $11111_2 = 2^3 + 2^2 + 2^1 + 2 + 1 = 63.$

3. Can you rewrite the sum $1 + 2 + 2^2 + 2^3 + \ldots + 2^{n-1}$ as a shorter formula?

Solution: $1 + 2 + 2^2 + 2^3 + \ldots + 2^{n-1} = 2^n - 1$ because the number $1 + 2 + 2^2 + 2^3 + \ldots + 2^{n-1} = 111\ldots1_2$ (with n ones) comes right before $1000 \ldots 0_2$ (with n zeroes) which is 2^n.

4. Binary Guessing Game:

a) Build a binary tetrahedron using the net on the next page and look out for patterns:
i) on the vertices ii) on each edge iii) on the faces

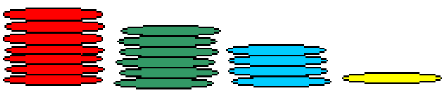
b) For each vertex, write down all the numbers connected to that vertex by 1 segment.
Describe a defining rule for each of these sets. Here they are in binary:
A: 1; 3; 5; 7; 9; 11; 13; 15
B: 2; 3; 6; 7; 10; 11; 14; 15
C: 4; 5; 6; 7; 12; 13; 14; 15
D: 8; 9; 10; 11; 12; 13; 14; 15

If I pick a number between 1 and 15 and tell you exactly in which of the sets above it is, can you tell me the number and its binary form without looking at the tetrahedron? (you can try by looking first).

This will be assigned as homework.

2. Nim with piles

Nim is a game of strategy. There are many variants but we will try this one: Start with any number of counters in any number of piles. Two players take turns to remove any number of counters from a single pile. The winner is the player who takes the last counter.



We call a starting position of a game a WIN position if there is a strategy by which the first player can win. We call it a LOSE position if there is a strategy for the second player to win the game against the first player. In each of the following starting position, decide whether it is a winning or a losing position:

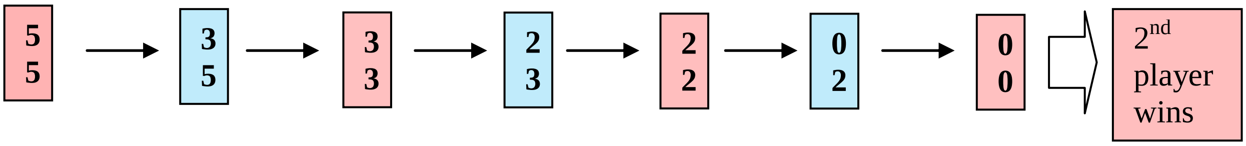
a) Playing with two piles:

Counters in each pile	1st player wins	2nd player wins
1, 1		V
1, 2	V	
2, 2		V
2, 5	V	
3, 3		V

The positions in which the 2nd player wins are those in which the two piles have the same numbers of counters, because whatever move the 1st player makes with one pile, the 2nd player can always **mirror** with the other pile.

Example:

Start 1st player 2nd player 1st player 2nd player 1st player 2nd player

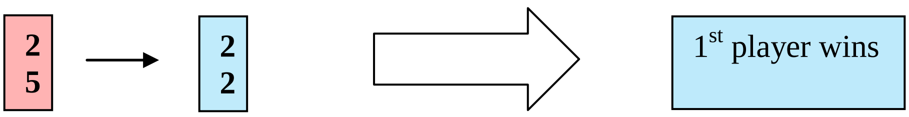


Note: For these positions, the winner always has the same colour as the starting position. Thus colour-coding the players is a very handy way to keep track of who will win.

If you start with any other position you can always take away the extra counters from the larger pile, thus making the piles even.

Example:

Start 1st player



First player has brought the game to a position which is a win for the current 2nd player, but after the first move, the 1st player has become 2nd, so he/she WINS.

b) Playing with three piles:

Counters in each pile	1st player wins	2nd player wins
1, 1, 3	V	
1, 2, 2	V	
1, 2, 3		V
1, 3, 4	V	
1, 4, 5		

Starting with the smaller numbers and moving on to bigger ones, take any 3 numbers on the binary tetrahedron.

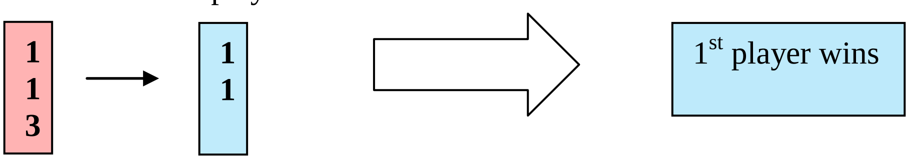
i) Who will win if the 3 numbers are all on the same line?

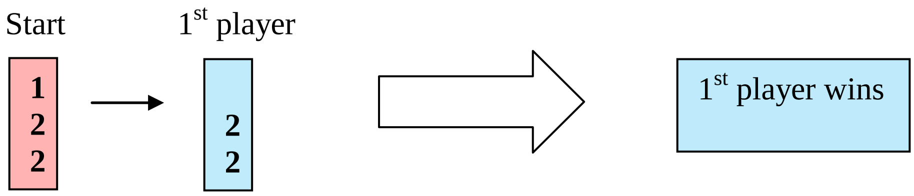
ii) Who will win if the 3 numbers include 2 vertices but are not all on the same line?

iii) Find more starting positions which insure that the 1st player loses.

The first two positions in the table are WINs because the 1st player can leave the 2nd player with 2 piles of equal numbers of counters:

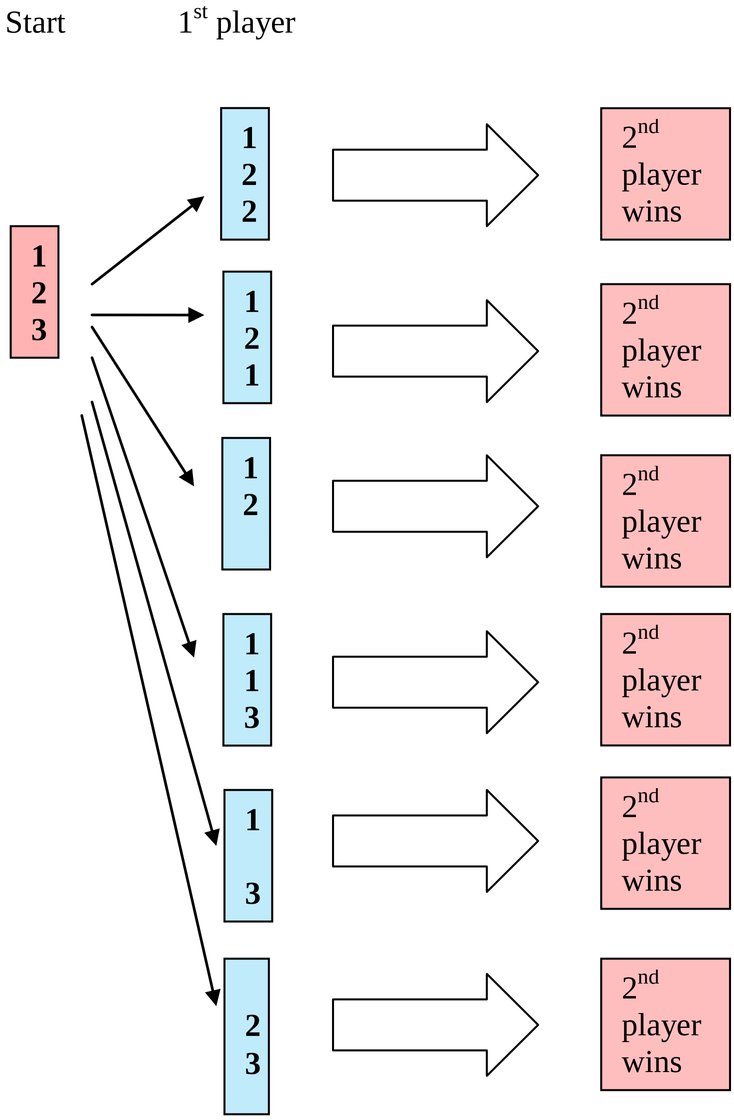
Start 1st player



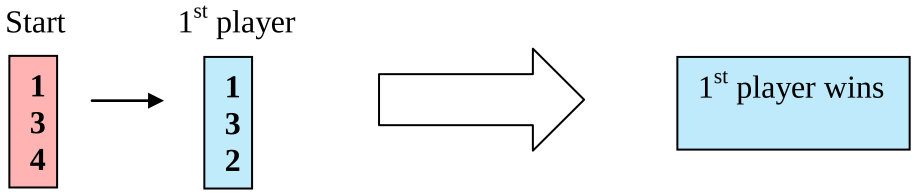


Note that in these cases, the winner is coloured differently from the starting position.

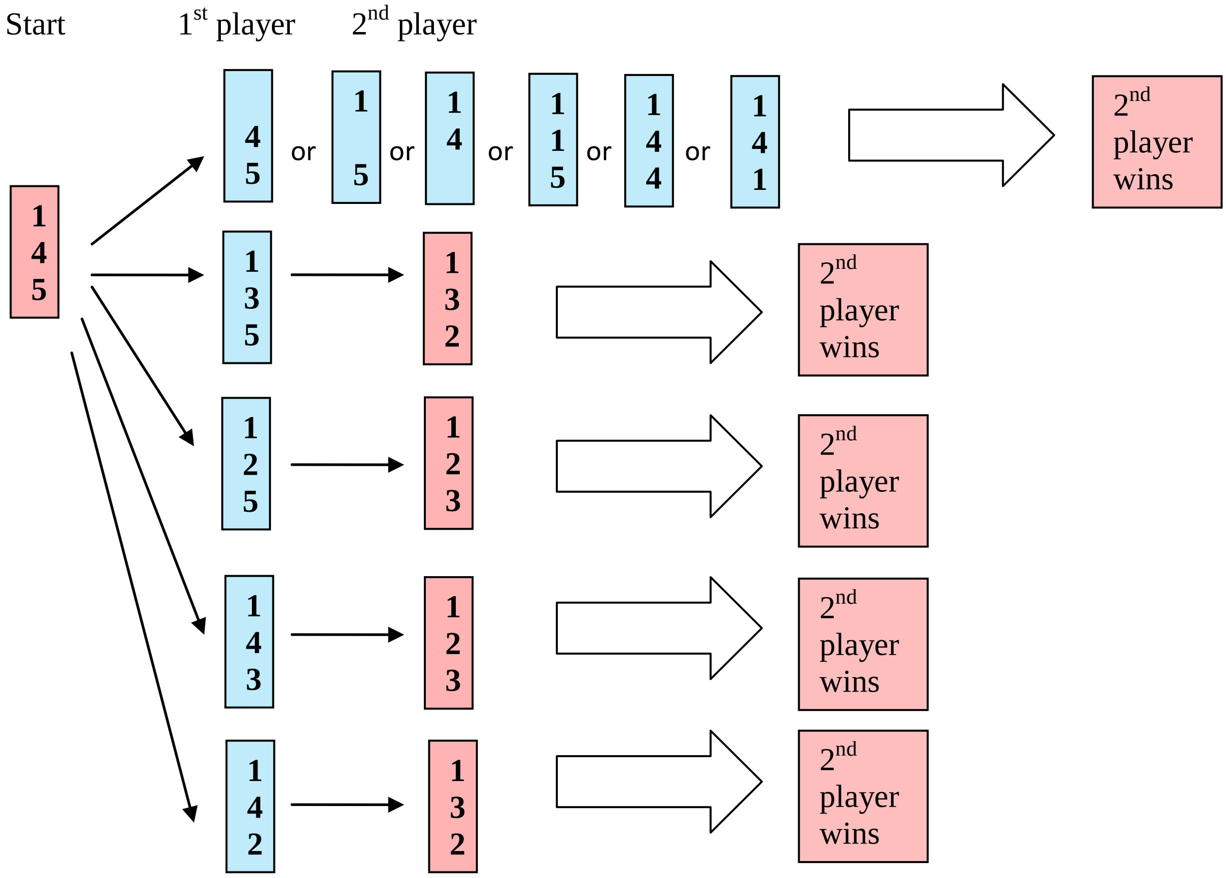
The starting position 1,2,3 leads the 2nd player winning the game. To prove this, we have to show that all possible moves of the 1st player lead him to a position where the other player wins:



Position 1, 3, 4 leads to the 1st player winning because it can be sent to 1, 3, 2 in one move.



Position 1, 4, 5 is a LOSE because whatever the 1st move, the 2nd move can send the game to one of these LOSE positions: 1,2,3 (or a permutation) or 4,4, or 1,1.



i) If the 3 numbers are all on the same line in the tetrahedron, then the 2nd player wins.

Indeed, starting with 3 numbers on the same line and changing one of them with a smaller number, this smaller number, together with one of the other 2 initial numbers, form a new line in the tetrahedron. Now the remaining of the 3 initial numbers can be swapped with a number on the new line, which is always smaller than the original one. Indeed, this can be checked algebraically because any 3 collinear numbers a, b, c satisfy (possibly after reordering): $b = a + c$. If the 1st move is

$a, b, c \ \rightarrow a, b, c'$ and $c' < c$ then $b' = a + c' < a + c = b$ so the next move can be $a, b, c' \rightarrow a, b', c'$. Now a, b', c' are all on the same line, but are smaller than the starting numbers. We can continue this way till reaching 1,2,3 which is a LOSE for the 1st player.

If the 1st move is

$a, b, c \ \rightarrow a, b', c$ and $b' < b$ then $c' = b' - a < b - a = c$ so the next move can be $a, b', c \rightarrow a, b', c'$.

ii) If the 3 numbers include 2 vertices but are not all on the same line, then they form a WIN position.

Indeed this move can be reduced in 1 move to a position where the 3 numbers are all on the same line.

Algebraically, if a, b, c with $a < b < c$ satisfy $b > a + c$ then take $a, b, c \rightarrow a, a + c, c$ and if $b < a + c$ then take $a, b, c \rightarrow a, b, b - a$.

In fact all other 3 points not on a line can boast the same property except some triples of midpoints.

iii) Can you find other LOSING positions?

There are some triplets of midpoints which also form LOSE positions. That is because these triplets a, b, c also satisfy $b = a + c$ as if they were collinear. You can check on the tetrahedron that the only moves available lead to WIN positions above.

(Note to tutors: the “binary tetrahedron” is in fact secretly the 3 dimensional projective spaces with coordinates in Z_2, known as the Fano projective space. In this space, all midpoints of edges form a plane and the triplets of midpoints mentioned above are collinear, indeed.)

c) Playing with four piles:

c) Playing with four piles:

Counters in each pile	1st player wins	2nd player wins
1, 1, 2, 2		V
1, 1, 2, 3	V	
m,m,n,n		V

Using the binary tetrahedron above, can you find more 4 pile positions which insure that the 1st player will lose?
How about when the 1st player wins?

WIN positions: m,m,n,p with p different from n.
LOSE positions: 1, 3, 5, 7 and similar positions in the tetrahedron. In 2 moves these can be reduced either to m,m,n,n or to a LOSE position with 3 points on a line like in b). No exhaustive proof should be required during classtime.

d) Look at all the positions discovered in the steps above in which the 2nd player wins. Write the numbers of counters in each pile in a column and convert them to binary. Do you notice any patterns?

Examples:

$$\begin{array}{c}1\\1\end{array}\qquad \begin{array}{l}2=10_2\\2=10_2\end{array}\qquad \begin{array}{l}3=11_2\\3=11_2\end{array}\qquad \begin{array}{l}5=101_2\\5=101_2\end{array}\qquad \begin{array}{l}1=\ 01_2\\2=\ 10_2\\3=\ 11_2\end{array}\qquad \begin{array}{l}1=001_2\\4=100_2\\5=101_2\end{array}\qquad \begin{array}{l}1=001_2\\3=011_2\\5=101_2\\7=111_2\end{array}$$

They all have even numbers of 1-s on each column.
In fact, this is always true: the 2nd player always wins when there are an even numbers of 1’s in each column of the binary codes for the starting position, because:
- the 1st player’s move will always disturb this property,
- but this property can then always be restored in one move.

$$\begin{array}{l}\mathbf{1=001_2}\\\mathbf{3=011_2}\\\mathbf{5=101_2}\\\mathbf{7=111_2}\\\hline \text{1's in columns:}\\\mathbf{2\ 2\ 4}\end{array}\rightarrow\begin{array}{l}\mathbf{1=001_2}\\\mathbf{3=011_2}\\\mathbf{5=101_2}\\\mathbf{4=100_2}\\\hline \text{1's in columns:}\\\mathbf{2\ 1\ 3}\end{array}\rightarrow\begin{array}{l}\mathbf{1=001_2}\\\mathbf{0=000_2}\\\mathbf{5=101_2}\\\mathbf{4=100_2}\\\hline \text{1's in columns:}\\\mathbf{2\ 0\ 2}\end{array}\rightarrow\begin{array}{l}\mathbf{1=001_2}\\\mathbf{0=000_2}\\\mathbf{2=010_2}\\\mathbf{4=100_2}\\\hline \text{1's in columns:}\\\mathbf{1\ 1\ 1}\end{array}\rightarrow\begin{array}{l}\mathbf{1=001_2}\\\mathbf{0=000_2}\\\mathbf{2=010_{22}}\\\mathbf{3=011}\\\hline \text{1's in columns:}\\\mathbf{0\ 2\ 2}\end{array}$$

Other Number Bases

1. Here are a few:

Number Base	Uses symbols:	Counts in powers of:
Ternary	0, 1, 2	3
Quaternary	0, 1, 2, 3	4
Octal	0, 1, 2, 3, 4, 5, 6, 7	8
Hexadecimal	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F	16

2. Set the counter on the following webpage to different number bases. Stop the counter at random and figure out the number reached in decimal notation.
http://www.mathsisfun.com/binary-decimal-hexadecimal.html
Have fun?

3. Because 8 and 16 are powers of 2, they are also convenient ways to package information stored in computers. Convert these binary numbers to the required bases:

(i) $0\,0\,1\,0\,1\,0\,0\,1\,1\,0\,0\,0\,1\,0\,0\,1\,1\,0\,1\,1\,1_2$ into octal.

(ii) $0\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,1\,0\,0\,0\,1\,0\,0\,1\,1\,0\,1\,0\,1\,1\,0\,1\,1\,1\,1\,1_2$ into hexadecimal.

Solution: Using our insights into binary numbers: $8 = 2^3$ so if we group the digits of the binary numbers into groups of 3, each group will give us an octal digit.
To start with a simple example,

$$110111_2 = 110000_2 + 111_2 = 110_2 \times 1000_2 + 111_2 = 6 \times 8 + 7 = 67_8$$

Note that $8^2 = 2^6$, $8^3 = 2^9$, $8^4 = 2^{12}$ etc which explains why our procedure works with more groups of 3 as well. You might go through $101\ 110\ 111_2$ step by step as before.
Similarly, for hexadecimal numbers we work in groups of 4 because $2^4 = 16$, $2^8 = 16^2$, $2^{12} = 16^3$ etc.

The binary number:	001	010	011	000	100	101	110	111
The octal number:	1	2	3	0	4	5	6	7

The binary number:	0001	0010	0100	1000	1001	1010	1101	1111
The hexadecimal number:	1	2	5	8	9	A	D	F

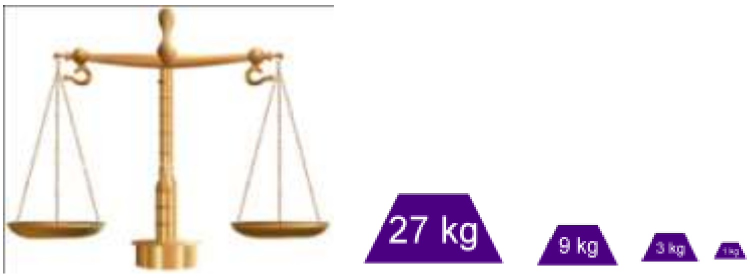
b) Why do programmers always mix up Halloween and Christmas?

Solution: Because Oct 31 == Dec 25!

d) What is $777777 + 1$ in octal? Translate the equation in decimal.

4. Weighing Game:

You are given weighing scales and exactly 4 weights of 1, 3, 9 and 27kg, like in the picture.



a) Using these, can you make the following measurements: (i) 2kg (ii) 6kg (iii) 18kg (iv) 24kg
b) What are all the possible weight measurements you can make with the above?
c) Repeat the problem with the weights of 1, 1, 5, 5, 25, 25 and 125kg.

Solution: a)2=3-1. If we place the 3kg weight on the right scale, and the 1kg weight together with the quantity to be measured on the left scale, we can measure 2kg.
Similarly, 6=9-3, 18=27-9, 24=27-3

b) The largest weight we can measure is 1+3+9+27=40. This is 1111_3. All the numbers between 1 and 40 written as ternary numbers are 1, 2, 10, 11, 12, 20, 21, 22, 100, 101, 102, ..., 1111. We need to write them using only the ternary numbers 1, 10, 100 and 1000, each at most once, and the operations + and -.
We already saw how the ternary numbers 2=10-1, 20=100-10, 200=1000-100. We can add any combination of these, and also with 1, 10, 100. The resulting expressions get to use the ternary numbers 1, 10, 100 and 1000, each at most once, with + and -.
For example, 121=100+20+1=100+100-10+1=200-10+1=1000-100-10+1 which in decimal numbers shows 16=27-9-3+1.
1022=222+100=1000-1+100 translates in decimals: 35=27+9-1.
Finally, ternary numbers like those between 1100 and 1111 are formed by adding 1, 10, 100 and 1000, each at most once.

c) The same principle applies here: We write any number between 1 and $187 = 1222_5$ in base 5 and use

$$3 \times 5 = 5^2 - 2 \times 5, \qquad 4 \times 5 = 5^2 - 5$$
$$3 \times 5^2 = 5^3 - 2 \times 5^2, \qquad 4 \times 5^2 = 5^3 - 5^2$$

Number Bases and Polynomials

1. Mystery Basis:

An evil king wrote three secret two-digit numbers $a;\ b;\ c$. A handsome prince must name three numbers $X;\ Y;\ Z$, after which the king will tell him the sum $aX + bY + cZ$. The prince must then name all three of the King's numbers, or he will be executed. Help out the prince!

Solution: We work in basis 100: Let $a = 1, b = 100$ and $c = 10000$.

2. Best basis:

a) Translate this decimal number addition in a more suitable basis. Then translate the answer back to decimals.

$(i)\ 5(1+6+6^2+\ldots+6^{n-1})+1$ $(ii)\ 6\,(1+7+7^2+\ldots+7^{n-1})+1$

b) Can you write $X^n -1$ as a product of two polynomials? Explain why this works.

Solution: In base 6 we have $555\ldots5_6+1_6=1000\ldots0_6$ which translates as

$$5(1+6+6^2+\ldots+6^{n-1})+1=6^n$$

And similarly for (ii).

b) If X was a positive integer then working in base X as before gives $(X-1)(1+X+\cdots+X^{n-1})+1=X^n$ so $(X-1)(1+X+\cdots+X^{n-1})=X^n-1$ which can be verified by multiplying the terms in the brackets using distributivity.

3. Any basis?

Having been abducted by aliens from the exoplanet Xari, after an extraordinary journey I was confronted by an extremely angry court martial who accused my species of priding itself with excessive knowledge of prime numbers. There was only one way to redeem myself and that was to determine whether **2323** is a prime: a question they have been stumped with for too long. The question seemed simple enough, however I was terrified of getting it wrong, because I had no idea what number base they were using!

Still, after thinking for a minute, I confidently stated that 2323 is not prime, as $2323=23\times 101$.

They were happy with the answer so they let me go.

a) Was I just plain lucky, or does this work in any number basis using the symbols 0,1,2,3? Does it matter if they read their numbers left-to-right or right-to-left?

b) Are these numbers primes or composites on Xari:

$(i)1224?$ $(ii)1001?$ $(iii)\ 2332?$

I still have no idea of the number basis on Xari, but the numbers above exist there as such.

Solution: a) In base X, this becomes $2X^3+3X^2+2X+3=(2X+3)(X^2+1)$. This can be verified directly by multiplication or by grouping terms and factoring:

$2X^3+3X^2+2X+3=X^2(2X+3)+2X+3=(X^2+1)(2X+3)$.

Even if the numbers are read from right-to-left the factorization works:

$2+3X+2X^2+3X^3=(2+3X)(1+X^2)$.

It may be worth noting the relation between $P(X)=2X^3+3X^2+2X+3$ and

$$Q(X)=2+3X+2X^2+3X^3=X^3P(\frac{1}{X})$$

b) We may verify in base 10, and then in any basis X, that these numbers are composites:

$(i)1224=12\times 102$ and $X^3+2X^2+2X+4=(X+2)(X^2+2)$.

$(ii)1001=11\times 91=11\times 7\times 13$ in base 10. Making sense of $7\times 13=91$ in other bases might be a bit tricky, but the factor 11 suggests trying to decompose:

$$X^3+1=(X+1)(?)$$

Which after some long division

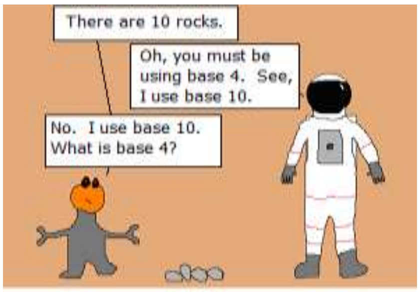
$$X^3+1=(X+1)(X^2-X+1)$$

Indeed when $X=10$ we get $10^2-10+1=91$.

$(iii)2332=2002+330=2\times 11\times 91+3\times 11\times 10=11\times(182+30)=11\times 212$. In general

$$2X^3+3X^2+3X+2=2(X^3+1)+3(X^2+X)=2(X+1)(X^2-X+1)+3(X+1)X$$
$$=(X+1)(2X^2-2X+2+3X)=(X+1)(2X^2+X+2)$$

4. More Aliens



The Alien story was inspired by the following link.
Try the last puzzle there:
http://www.artofproblemsolving.com/Forum/blog.php?b=65

Solution included on the webpage.

5. Mystery polynomials

a) Find a polynomial with nonnegative integer coefficients such that $P(2) = 100$ and $P(1) = 3$. Prove that there is only one such polynomial.

b) Suppose $P(x)$ is an unknown polynomial, of unknown degree, with nonnegative integer coefficients. You have access to an oracle that, given an integer m, spits out $P(m)$, the value of the polynomial at m. However, the oracle charges a fee for each such computation, so you want to minimize the number of computations you ask the oracle to do. Show that it is possible to uniquely determine the polynomial after only two consultations of the oracle.

Note: Coefficients are the numbers that occur in a polynomial as factors of the various powers of X, *including of* $X^0 = 1$.

Solution: a) Write 100 in base 2: $100 = 1100100_2$. So $P(X) = X^6 + X^5 + X^2$ satisfies $P(2) = 100$ and, incidentally, $P(1) = 3$. Moreover, this is the only such polynomial with coefficients 0 and 1 due to the unique way in which a number can be expressed as a binary number.

Now to show that the only solution is the one found above. Indeed, if $Q(X) = a_n X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0$ is a solution with non-negative integer coefficients $a_n, a_{n-1,} \dots, a_0$, then we can write each coefficient a_k in base 2. This amounts to writing $a_k = A_k(2)$ where $A_k(X)$ is a polynomial with coefficients 0 and 1. Define $S(X) = A_n(X)X^n + A_{n-1}(X)X^{n-1} + \cdots + A_1(X)X + A_0(X)$.
Then $S(2) = Q(2)$ =100 and $S(X)$ has coefficients 0 and 1 so it must be that $S(X) = X^6 + X^5 + X^2$. On the other hand

$$\begin{aligned} S(1) = A_n(1) + A_{n-1}(1) + \cdots + A_1(1) + A_0(1) &\le A_n(2) + A_{n-1}(2) + \cdots + A_1(2) + A_0(2) \\ &= a_n + a_{n-1} + \cdots + a_1 + a_0 = Q(1) \end{aligned}$$

And equality only holds if all coefficients a_k are 0 or 1.

b) Suppose $P(X) = a_n X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0$. If $X = m$ was a number larger than all coefficients, then writing $P(m)$ in base m would give us all the coefficients because

$$P(m) = a_n m^n + a_{n-1}m^{n-1} + \cdots + a_1 m + a_0.$$

So now how to find m larger than all the coefficients? Ask for $P(1)$ and once you know it, take $m = P(1) + 1$. Indeed, $P(1) + 1$ is larger than all the coefficients of $P(X)$.