Jumping Jiving GCD

Summary

This is a cycle of Lessons dedicated to GCD and LCM.
The first lesson introduces GCD and looks at methods of calculating it. Next it proves a surprising property of GCD and uses it to prove the Unique Prime Factorisation. This last part may be more difficult to absorb and can be skipped/postponed if students are not curious about the Unique Prime Factorisation.

The last part introduces LCM and looks at some methods of calculating it.

Resources

- 1 Question Sheet per student.
- 1 Lesson Outline – 3 pages per student

Questions/Suggestions?

If you plan to use this material, or if you would like to send feedback, please email
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Jumping Jiving GCD – Class outline.

The Greatest Common Divisor (GCD) a.k.a. Highest Common Factor (HCF)

The *greatest common divisor* of two numbers is the largest number that divides them both. The greatest common divisor of a and b is usually denoted by gcd (a, b).

Co-prime numbers

Exercise 1. Suppose that $\frac{a}{b}$ is the simplest form of a fraction. What is $\gcd(a, b)$?

If $\gcd(a, b) = 1$ then a and b are said to be *co-prime.* In other words, they have no other common divisors except 1.

Finding $\gcd(a, b)$:

Prime factorization.

Rule: $\gcd(a, b)$ = the product of all common primes, each chosen with its smallest index between the prime factorizations of a and b .

Exercise 2. a)Find $\gcd(345,\ 184)$.

b)Find $\gcd(a, b)$ for $a = 2^3 \times 3^2 \times 7 \times 13^5$, and $b = 2 \times 3^3 \times 5 \times 7^4$.

c) Find $\gcd(a, b, c)$ when $a = 2 \times 3 \times 4 \times 5$, and $b = 6 \times 7 \times 8 \times 9$ and $c = 10 \times 11 \times 12 \times 13$.

Euclid's algorithm.

Simplifying principles:

1) If a fraction $\frac{a}{b}$ can be simplified by a number, then $\frac{a}{b} - 1$, $\frac{a}{b} - 2$, $\frac{a}{b} - 3$, ... as well as $\frac{a}{b} + 1$, $\frac{a}{b} + 2$, $\frac{a}{b} + 3$ ... can be simplified by the same number.

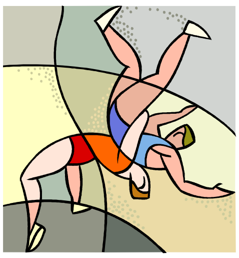
Example: $\frac{24}{16} = \frac{3}{2} = 1\frac{1}{2}$. Or, $\frac{24}{16} = 1\frac{8}{16} = 1\frac{1}{2}$. In both cases we simplified by 8.

2) If a fraction $\frac{a}{b}$ can be simplified by a number, then $\frac{b}{a}$ can be simplified by the same number.

Euclid's algorithm

Exercise 3: a) Find $\gcd(a, b)$ for $a = 23456$, and $b = 34567$.

b) Find $\gcd(a, b)$ for $a = 6n + 10$ and $b = 8n + 11$ where n is some unknown natural number.



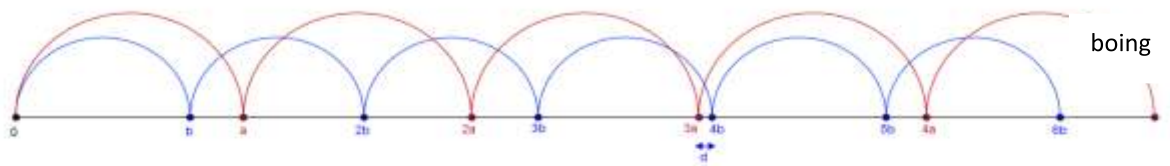
Memo Note: **Euclid's algorithm** closely resembles Greco-Roman wrestling between the numerator and the denominator:

1) First the denominator knocks a few units off the numerator.
2) Then it topples the numerator down and gets on top.
3) They start all over again.

A surprising property of the GCD.

A flea problem

Two fleas, one red and one blue, are jumping happily along a straight line. The red flea's jump length is $a = 17$ cm and the blue flea's jump length is $b = 13$ cm. They both start up from the same point 0 and jump at their leisure at varying speeds.
a) Find all the spots on the line where the two fleas can land in their first 5 steps, if they jump Eastward.
b) What is the closest from each other they can ever be on their line, without being on the same spot?
c) Repeat the problem in the case when $a = 34$ cm and $b = 26$ cm.
d) Repeat the problem in the case when $a = 51$ cm and $b = 39$ cm.



A blackboard game:

The following four symbols are written on a blackboard:

$$165, \qquad 120\,, \qquad +, \qquad -$$

A group of friends are playing the following game: They form new numbers by using any of the symbols in the list any number of times, but no other symbols.
a) How many numbers can they get this way?
b) What is the smallest positive number they can get in this way?
c) Think about all the numbers they can get this way. What do they all have in common?

The blackboard game with mystery numbers:

The following four symbols are written on a blackboard:

$$a, \qquad b\,, \qquad +, \qquad -$$

Where a and b are some mystery numbers, natural.
A group of friends are playing the following game: They add new numbers to the list by using any of the symbols in the list any number of times, but no other symbols.
Let d be the name of the smallest positive number they can get in this way.
a) Prove that all the numbers in the list are divisible by $\gcd(a, b)$.
b) Prove that $d|a$. Similarly, $d|b$.
c) Prove that $d = \gcd(a, b)$.

Property of the GCD:

The smallest positive number of the form $ax + by$ where x and y are random integers is $\gcd(a, b)$.

$\gcd(a, b)$ divides all numbers of the form $ax + by$ where x and y are integers.

Applications of the GCD property.

The Unique Prime Factorisation proved!

The Unique Prime Factorisation of integers comes from an important propery of primes:

a) Let p be a prime. If $p \mid ab$, then $p \mid a$ or $p \mid b$.

b) Let p be any number. If $p \mid ab$ and p is co-prime with b, then $p \mid a$.

A money problem

In a country all money are of two types: Golden coins worth 115 EU each and Silver coins worth 45 EU each.

a) Which prices can they use in this country and why?

b) You want to buy an item costing 5 EU. What are all the ways this transaction be made?

c) How about for an item costing 30 EU? For an item costing 32 EU?

Diophantine equations

Find integer solutions for the following equations or explain why they can't be found:

a) $345x + 123y = 50$

b) $345x + 123y = 51$.

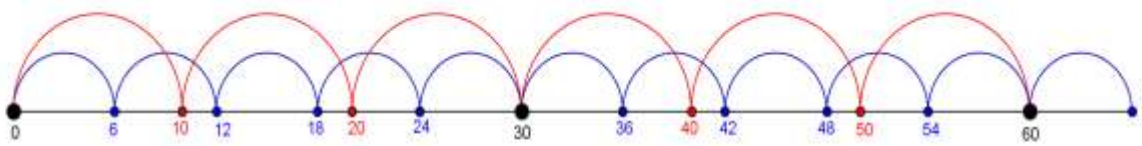
Last but Least LCM

The least flea problem

Two fleas, one red and one blue, are jumping happily Eastward along a straight line. The red flea's jump length is $a = 10$ cm and the blue flea's jump length is $b = 6$ cm. They both start up from the same point 0.

a) Find the point on the line closest to 0 where both fleas can land.

b) Describe all the spots on the line where both fleas can land.



The LCM Rule:

$$lcm(a, b) = \frac{ab}{\gcd(a, b)}$$

Mystery numbers

In each case, find all possible pairs of two mystery numbers with the stated propery:

a) their gcd is 4 and their lcm is 360.

b) their gcd is 15 and their lcm is 180.

Rule: $\operatorname{lcm}(a, b)$ = the product of all primes of a and b, each with its highest index between the prime factorizations of a and of b.

Find the least common multiple of a and b if

i) $a = 30$ and $b = 42$.

ii) $a = 2000$ and $b = 616$.

Jumping Jiving GCD – Class notes

1. Motivation: Simplifying Fractions

We know that there are many different ways of writing the same number as a fraction, for example:

$$\frac{2}{3} = \frac{50}{75}$$

Which means that carving out 2 thirds of a pie gives the same result as carving out the pie in 75 little pieces and picking 50 of them ... But the second method is hard work!

Based on this, we say that there is a “best” or easiest way to write a fraction, when it is impossible to cancel out anything from the numerator and the denominator. The fraction is said to be *reduced*, or in *simplest form*, in this case.

To bring a fraction to its simplest form one may do successive simplifications:

$$\frac{50}{75} = \frac{50 \div 5}{75 \div 5} = \frac{10 \div 5}{15 \div 5} = \frac{2}{3}$$

Or one can do it in one strike like this:

$$\frac{50}{75} = \frac{50 \div 25}{50 \div 25} = \frac{2}{3}$$

In both cases one simplifies by numbers that divide both the numerator and denominator.

The Greatest Common Divisor (GCD) a.k.a. Highest Common Factor (HCF)

Question: What is special about the number 25 above? (in relation to 50 and 75).

Answer: 25 is the largest natural number which divides both 50 and 75.

The *greatest common divisor* of two numbers is the largest number that divides them both.

The greatest common divisor of a and b is usually denoted by $\gcd(a, b)$.

For example, $\gcd(50, 75) = 25$.

Co-prime numbers

Question:

Suppose that $\frac{a}{b}$ is the simplest form of a fraction.

What is $\gcd(a, b)$?

Answer:

$\gcd(a, b) = 1$.

If $\gcd(a, b) = 1$ then a and b are said to be *co-prime.* In other words, they have no other common divisors except 1.

Question:

Can you find a number which is co-prime with all numbers smaller than itself?

Can you name all such numbers?
Answer:
The prime numbers.

Finding $\mathbf{gcd}(\boldsymbol{a}, \boldsymbol{b})$.

Take any two numbers a and b. Next we'll look at different methods for finding $\gcd(a, b)$, that is, of bringing a fraction $\frac{a}{b}$ to its simplest form.

Method I: Successive simplification. Looking back at our first example

$$\frac{50}{75} = \frac{50 \div 5}{75 \div 5} = \frac{10 \div 5}{15 \div 5} = \frac{2}{3}$$

How can $\gcd(50, 75)$ be extracted from this calculation?

Answer:
$\gcd(50,75) = 25 = 5 \times 5$, the product of the numbers by which we simplified.
That's because $50 \div 5 \div 5 = 50 \div (5 \times 5) = 50 \div 25 = 2$ and
$75 \div 5 \div 5 = 75 \div (5 \times 5) = 70 \div 25 = 3$ and 2 and 3 clearly have no other common divisors.

Practice exercise: Use successive simplification to find $\gcd(60,84)$.
Answer: (in its goriest form):

$$\frac{60}{84} = \frac{60 \div 2}{84 \div 2} = \frac{30 \div 2}{42 \div 2} = \frac{15 \div 3}{21 \div 3} = \frac{5}{7}$$

We simplified by 2, 2, and 3 so $\gcd(60,84) = 2 \times 2 \times 3 = 12$.

Method II: Prime factorization.
This worked well and good, but what if we can't easily find any common divisors?

Example: Find $\gcd(345, \ 184)$.
Solution: We can't find a number that divides both 345 and 184 right away, but we can find some divisors for each of them at a time. So let's just prime factorize the numbers.
$\mathbf{345 = 5 \times 3 \times 23}$ and $\mathbf{184 = 2^3 \times 23}$.
And $\gcd(345, 184) = 23$.

Now suppose we have prime factorized two numbers. Let's derive some basic rules for finding their gcd.

Rule: $\gcd(a, b)$ = the product of all common primes, each chosen with its smallest index between the prime factorizations of a and b.

Example: Find $\gcd(a, b)$ for $a = 2^3 \times 3^2 \times 7 \times 13^5$, and $b = 2 \times 3^3 \times 5 \times 7^4$.
Solution: $\gcd(a, b) = 2 \times 3^2 \times 7 = 126$.

Explanation: The prime 2 comes up with index 3 in a and index 1 in b, so in $\gcd(a, b)$ it will come up with index 1= min(3, 1). Recall that $2 = 2^1$.
Indeed, 2 divides both a and b but 2^2 wouldn't divide b so it can't be a factor of $\gcd(a, b)$.

We can generalize to as many numbers we want.

Practice Exercise: Find $\gcd(a, b, c)$ when $a = 2 \times 3 \times 4 \times 5$, and $b = 6 \times 7 \times 8 \times 9$ and $c = 10 \times 11 \times 12 \times 13$.
Solution: Prime factorisation:
$a = 2^3 \times 3 \times 5$ and $b = 2^4 \times 3^3 \times 7$ and $c = 2^3 \times 3 \times 5 \times 11 \times 13$.
So $\gcd(a, b, c) = 2^3 \times 3 = 24$.

Method III: Euclid's algorithm.

What if even prime factorization fails with numbers a and b: what if a and b are too large to factorize, or what if they are written as sums with terms depending of an unknown number n?

Strategy: reduce our numbers to manageable proportions, using these simple

Simplifying principles:
1) If a fraction $\frac{a}{b}$ can be simplified by a number, then $\frac{a}{b} - 1$, $\frac{a}{b} - 2$, $\frac{a}{b} - 3$, ... as well as $\frac{a}{b} + 1$, $\frac{a}{b} + 2$, $\frac{a}{b} + 3$ ... can be simplified by the same number.
Example: $\frac{24}{16} = \frac{3}{2} = 1\frac{1}{2}$. Or, $\frac{24}{16} = 1\frac{8}{16} = 1\frac{1}{2}$. In both cases we simplified by 8.

2) If a fraction $\frac{a}{b}$ can be simplified by a number, then $\frac{b}{a}$ can be simplified by the same number.

Proof of the principles:
Point 2) Is pretty clear, as in both cases a and b are divisible by that same number.
For 1), $\frac{a}{b} - 1 = \frac{a-b}{b}$. Let's assume $\frac{a}{b}$ can be simplified by a number d. That is, $d | a$ and $d | b$. But then $d | (a - b)$. So both numerator and denominator of $\frac{a-b}{b}$ are divisible by d.
We can adapt this argument for all the other fractions.

Euclid's algorithm for finding $\gcd(a, b)$, the magic simplifying factor of the fraction $\frac{a}{b}$:

Let's suppose that the fraction is what you call improper, so the denominator (bottom) is smaller than the numerator (top). Then

1) We divide the numerator by the denominator, then discard the integer parts.
2) We turn the resulting fraction upside down, to get its reciprocal.
3) The new fraction can be simplified by the same numbers as the original one, but it has smaller top and bottom. We apply steps 1) and 2) again as many times as necessary until the fraction becomes manageable.

Example: Find $\gcd(a, b)$ for $a = 23456$, and $b = 34567$.

Solution:
i) Since $34567 > 23456$, let's start with $\frac{34567}{23456}$:

$$\frac{34567}{23456} = 1 + \frac{11111}{23456}$$

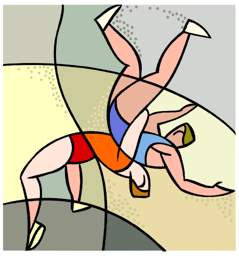
$$\frac{23456}{11111} = 2 + \frac{1234}{11111}\ .$$

$$\frac{11111}{1234} = 9 + \frac{5}{1234}$$

5 clearly doesn't divide 1234 so $\frac{5}{1234}$ is in reduced form, 5 and 1234 are co-prime.
This implies that $\frac{11111}{1234}, \frac{1234}{11111}, \frac{23456}{11111}, \frac{11111}{23456}$ and hence $\frac{34567}{23456}$ are all in reduced form, so 34567 and 23456 are co-prime. In terms of gcd-s we can write:

gcd(34567, 23456)=gcd(23456, 11111)=gcd(11111, 1234)=gcd(1234, 5)=1.

Memo Note: **Euclid's algorithm** closely resembles Greco-Roman wrestling between the numerator and the denominator:



1) First the denominator knocks a few units off the numerator.
2) Then it topples the numerator down and gets on top.
3) They start all over again.

This analogy seems quite appropriate because Euclid founded a famous Greek school of mathematics. But I doubt he enjoyed wrestling.

Example: Find $\gcd(a, b)$ for $a = 6n + 10$ and $b = 8n + 11$ where n is some unknown natural number.

Solution I

$a = 6n + 10$ and $b = 8n + 11$ where n is some unknown natural number.
We write $8n + 11 = 6n + 10 + 2n + 1$ so

$$\frac{8n + 11}{6n + 10} = 1 + \frac{2n + 1}{6n + 10}$$

Then $6n + 10 = 3(2n + 1) + 7$ so

$$\frac{6n + 10}{2n + 1} = 3 + \frac{7}{2n + 1}$$

We could even go on one more time like this:
$2n + 1 = 7 + 2n - 6$ so

$$\frac{2n + 1}{7} = 1 + \frac{2n - 6}{7}$$

Now we have two cases:

- For some values of n like 1, 2, 4, 5, 6, 7, 8, 9, 11,..., the number $2n - 6$ is not divisible by 7, so $2n - 6$ and 7 are co-prime. This makes a and b co-prime also.
- For other values of n like 3, 10, 17, 24, ... the number $2n - 6$ is divisible by 7, so gcd($2n - 6$,7)= gcd(a, b)=7. How do we recognize these values of n?
  Well, because $2n - 6 = 2(n - 3)$ and 7 is a prime, so if 7| $2(n - 3)$ then 7|$(n - 3)$, so $n - 3 = 7k$ where k can take the values 1,2,3,...

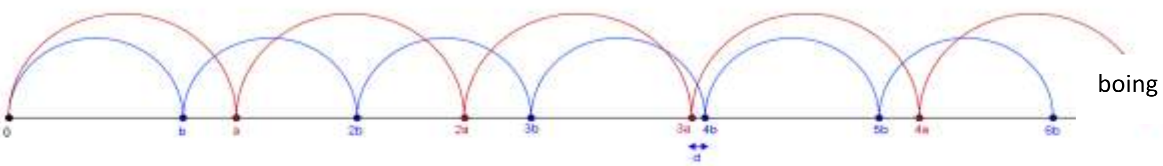
Solution II

We could have saved ourselves a lot of trouble if we decided to eliminate n right away:
$a = 6n + 10$ and $b = 8n + 11$.
There are 6 n-s in a and 8 n-s in b). (6 and 8 are called coefficients of n).
We notice that $6 \times 4 = 8 \times 3$ and so
$4a - 3b = 4(6n + 10) - 3(8n + 11) = 24n + 40 - 24n - 33 = 7$ no more n-s!
Let's denote gcd(a, b) by d. Now if $d|a$ and $d|b$ then $d|(4a - 3b)$, (prove this!)
so $d|7$ so d can only be 7 or 1. It remains to check that either of these values can occur, like in the first solution we used.

A surprising property of the GCD.

A flea problem

Two fleas, one red and one blue, are jumping happily along a straight line. The red flea's jump length is $a = 17$ cm and the blue flea's jump length is $b = 13$ cm. They both start up from the same point 0 and jump at their leisure at varying speeds.
a) Find all the spots on the line where the two fleas can land in their first 5 steps, if they jump Eastward.
b) What is the closest from each other they can ever be on their line, without being on the same spot?
c) Repeat the problem in the case when $a = 34$ cm and $b = 26$ cm.
d) Repeat the problem in the case when $a = 51$ cm and $b = 39$ cm.



Solutions:
a) Since the red flea can land every 17 cm, the spots it lands on are
$1 \times 17 = 17,\quad 2 \times 17 = 34,\quad 3 \times 17 = 51,\quad 4 \times 17 = 68,\quad 5 \times 17 = 85.$
Since the blue flea can land every 13 cm, the spots it lands on are
$1 \times 13 = 13,\quad 2 \times 13 = 26,\quad 3 \times 13 = 39,\quad 4 \times 13 = 52,\quad 5 \times 13 = 65.$

b) From the table above we see that $4b - 3a = 4 \times 13 - 3 \times 17 = 1,$
so the fleas can get within 1 cm of each other. They cannot get any closer without bumping into each other, because the places they land in are natural numbers and their smallest positive difference can only be 1.

c) In the case when $a = 34 = 17 \times 2$ cm and $b = 26 = 13 \times 2$ cm, all the measures of the landing spots above get doubled. Now $4b - 3a = 4 \times 26 - 3 \times 34 = 2$, so the fleas can get within 2 cm of each other. They can never get any closer than that without bumping into each other. Indeed, in this case the landing spots of the two fleas are all even, so the distances between them will always be even. This is due to the fact that $2 = \gcd(34, 26)$.

d) In the case when $a = 51 = 17 \times 3$ cm and $b = 39 = 13 \times 3$ cm, all the distances will be multiples of 3. Now $4b - 3a = 4 \times 39 - 3 \times 51 = 3$, so the fleas can get within 3 cm of each other, but they can never get any closer without bumping into each other. This is due to the fact that $3 = \gcd(51, 39)$.

The flea problem, mathematical formulation:

Let's discuss the case when $a = 17$ cm and $b = 13$ cm.

Then the red flea's landing spots are all the multiples of 17 cm. The blue flea's landing spots are all the multiples of 13 cm.

a) The landing spots of the fleas are multiples of the jump lengths.

b) $1 = \gcd(17, 13)$ is the smallest distance between any two multiples of 17 and 13.

c) $2 = \gcd(34, 26)$ is the smallest distance between any two multiples of 34 and 26.

d) $3 = \gcd(51, 39)$ is the smallest distance between any two multiples of 51 and 39.

A blackboard game:

The following four symbols are written on a blackboard:

$$165, \qquad 120\,, \qquad +, \qquad -$$

A group of friends are playing the following game: They form new numbers by using any of the symbols in the list any number of times, but no other symbols.

a) How many numbers can they get this way?

b) What is the smallest positive number they can get in this way?

c) Think about all the numbers they can get this way. What do they all have in common?

Solution:

a) They can get infinitely many numbers. In fact, just using 165 and + any number of times we get: $165 + 165 + \cdots + 165 = 165\,x$ for some natural number x. We can also get $165\,x$ for any negative integer number x, just by using $-165 - 165 - \cdots - 165$.
Similarly we can get all numbers $120\,y$ for any integer y if we use 120 and $+$ or $-$. All in all, we can get all numbers of the form $165x + 120y$ where x and y are integers. We can get no other numbers by playing with $165, \quad 120\,, \quad +, \quad -$.

b) Note that once you get a new number by the rules above, you can add it to the list of symbols and use it in your later calculations since it is written only in terms of $165, 120, +, -$.

For start, you can divide 165 by 120. The remainder can be written in terms of 165 and 120:
$165 = 1 \times 120 + 45$, so $45 = 165 - 120$.

To get a positive number smaller than 45, you can divide 120 by 45:
$120 = 2 \times 45 + 30$. Since 45 can be written only with $165, 120, +, -$, then so can 30:

$$30 = 120 - 45 - 45 = 120 - (165 - 120) - (165 - 120)$$
$$= 120 - 165 + 120 - 165 + 120$$

Now you have the numbers 165, 120, 45 and 30. To get a smaller number, divide 45 by 30: $45 = 1 \times 30 + 15$. Since 45 and 30 can be written only with $165, 120, +, -$, then so can 15:

$$15 = 45 - 30 = (165 - 120) - (120 - 165 + 120 - 165 + 120)$$
$$= 165 - 120 - 120 + 165 - 120 + 165 - 120.$$

More elegantly, we can write this as $15 = 3 \times 165 - 4 \times 120$.
So now we have the numbers 165, 120, 45, 30 and 15.
But now $30 = 2 \times 15$ with no remainder, so you can get no smaller positive numbers this way.

Note that $15 =$ gcd (165, 120). In fact, our calculations above follow Euclid's algorithm (written without fractions). We were able to write $15 = 165x + 120y$ for $x = 3$ and $y = -4$.

It looks like 15 is the smallest positive number we can get, but can we prove that no smaller positive number can be found?

c) All the numbers we get are of the form $165x + 120y = 15(11x + 9y)$ so they are all multiples of 15. Then indeed, we can get no positive numbers smaller than 15.

The blackboard game with mystery numbers:

The following four symbols are written on a blackboard:

$$a, \qquad b\,, \qquad +, \qquad -$$

Where a and b are some mystery numbers, natural.
A group of friends are playing the following game: They add new numbers to the list by using any of the symbols in the list any number of times, but no other symbols.
Let d be the name of the smallest positive number they can get in this way.
a) Prove that all the numbers in the list are divisible by $\gcd(a, b)$.
b) Prove that $d|a$. Similarly, $d|b$.
c) Prove that $d = \gcd(a, b)$.

Solution:
a) All numbers in obtained this way are of the form $ax + by$ so divisible by $\gcd(a, b)$.
b) If d didn't divide a then we would get a remainder which would also be on the list, but smaller than d. This is impossible since d is supposed to be the smallest positive number on the list.
c) d is a divisor of both a and b and it is divisible by $\gcd(a, b)$.

Property of the GCD:

The smallest positive number of the form $ax + by$ where x and y are random integers is $\gcd(a, b)$.

$\gcd(a, b)$ divides all numbers of the form $ax + by$ where x and y are integers.

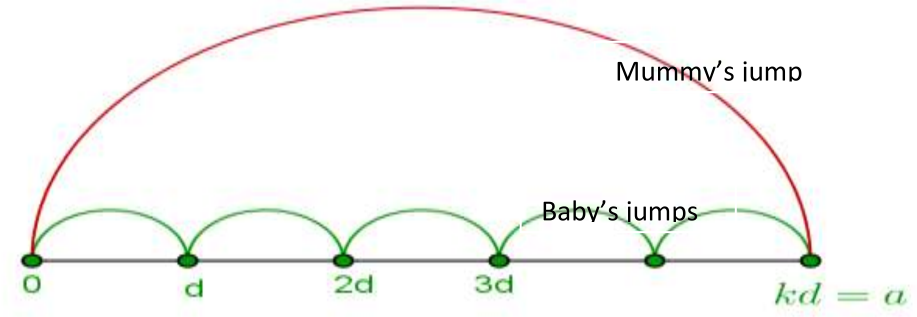
A generalized flea problem. Here a, b, d are natural numbers.

Mammy flea's jump is a cm long, Daddy's jump is b cm long, and Baby is, for the time being, piggybacking on Daddy.

Mammy and Daddy start from the same point and jump on a straight line until they get the closest they can be to each other (without sharing the same spot). Let's call this distance d cm. Then Baby flea discovers her jump is exactly the distance d between Daddy and Mammy. She then jumps happily away to play. Prove that Baby can reach Mammy in any spot Mammy lands. Prove that Baby can reach Daddy too.

Solution:

This is the same as the blackboard problem, just formulated in jumps rather than numbers.



Applications of the GCD property:

The Unique Prime Factorisation finally proved!

Recall that we reformulated the Unique Prime Factorisation as an Important propery of primes: *Let p be a prime. If $p \mid ab$, then $p \mid a$ or $p \mid b$.*

We can improve a little on this statement by:

Let p be any number. If $p \mid ab$ and p is co-prime with b, then $p \mid a$.

Proof:

We rely on the fact that p is co-prime with all numbers which are not its multiples. So assuming that p doesn't divide b, then

$1 = \gcd(b, p) = bx - py$ for some x and y integers. Multiply by a so that we get $a = abx - apy$. Now p divides both abx and apy, so p divides a.

Exercise: In a country all money are of two types: Golden coins worth 115 EU each and Silver coins worth 45 EU each.

a) Which prices can they use in this country and why?

b) You want to buy an item costing 5 EU. What are all the ways this transaction be made?

c) How about for an item costing 30 EU? For an item costing 32 EU?

Solution:

a) The buyers and sellers can use any multiples of 115 EU and 45 EU both as payments and change, so they can use prices of the form

$$115x + 45y.$$

As 5= gcd(115, 45)= the smallest number of this form, this is the smallest value which can be transacted. Any multiple of 5 can be transacted.

b) We can obtain 5 by successive divisions like in Euclid's algorithm. At each step, we write the new amounts we get in terms of 115 and 45:

$115 = 45 \times 2 + 25$ so $25 = 115 - 45 \times 2$.

$45 = 25 + 20$ so $20 = 45 - 25 = 45 - (115 - 45 \times 2) = 45 \times 3 - 115$.

$25 = 20 + 5$ so $5 = 25 - 20 = (115 - 45 \times 2) - (45 \times 3 - 115) = 115 \times 2 - 45 \times 5$.

So I could pay 2 Golden coins and get 5 Silver coins back.

Suppose that I pay 5 EU differently: $5 = 115x - 45y$. Then

$$115x - 45y = 115 \times 2 - 45 \times 5.$$

Separating multiples of 115 from those of 45:

$$115(x - 2) = 45(y - 5).$$

Divide by 5: $$23(x - 2) = 9(y - 5).$$

Now 9 must divide $23(x - 2)$ but it doesn't divide 23, so then 9 must divide $x - 2$:

$$x - 2 = 9k$$

For some integer k. Subing into $23(x - 2) = 9(y - 5)$ we get

$$y - 5 = 23k$$

Then $x = 9k + 2$ and $y = 23k + 5$ are the integer solutions for $5 = 115x - 45y$.

c) We now want to solve $30 = 115u - 45v$. We found

$$5 = 115 \times 2 - 45 \times 5$$

So if we multiply both sides by 6 we get

$$30 = 115 \times 12 - 45 \times 30.$$

If we proceed like in the previous case and look for other solutions

$$30 = 115u + 45v = 115 \times 12 - 45 \times 30,$$

then just as before we will get that $x = 9k + 12$ and $y = 23k + 30$ are the integer solutions for $30 = 115u - 45v$.

On the other hand, the equation $32 = 115u - 45v$ doesn't have solutions because 32 is not a multiple of 5 while the right-hand-side is.

Exercise: Find integer solutions for the following equations or explain why they can't be found:

a) $345x + 123y = 50$

b) $345x + 123y = 51$.

Solution:

a) We find gcd(345, 123):

$345 = 123 \times 2 + 99$ so $99 = 345 - 123 \times 2$.

$123 = 99 \times 1 + 24$ so $24 = 123 - 99 = 123 - (345 - 123 \times 2) = 123 \times 3 - 345$.

$99 = 24 \times 4 + 3$ so $3 = 99 - 24 \times 4 = (345 - 123 \times 2) - 4(123 \times 3 - 345)$

$$= 345 - 123 \times 2 - 123 \times 12 + 345 \times 4$$

Now 3 divides 24 and hence also 99, 123 and 345 and so $3 = 345 \times 5 - 123 \times 14$ is the gcd of 345 and 123, and it must divide all numbers of the form $345x + 123y$.
Hence no such numbers can be 50.

b) $51 = 3 \times 17 = (345 \times 5 - 123 \times 14) \times 17 = 345 \times 85 - 123 \times 238$,
Now for any solution of the equation $51 = 345x + 123y$ we have

$$345 \times 85 - 123 \times 238 = 345x + 123y,$$

$$345(85 - x) = 123(y + 238)$$

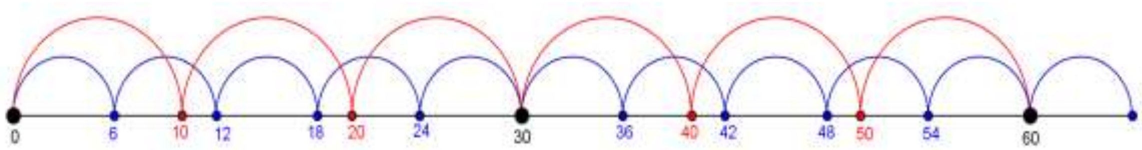
Divide both sides by 3: $\quad 115(85 - x) = 41(y + 238)$
Hence $85 - x = 41k$ and $y + 238 = 115k$.
So $x = 85 - 41k$ and $y = 115k - 238$.
Check: $345x + 123y = 345(85 - 41k) + 123(115k - 238) = 51$.

Last but Least LCM

The least flea problem

Two fleas, one red and one blue, are jumping happily Eastward along a straight line. The red flea's jump length is $a = 10$ cm and the blue flea's jump length is $b = 6$ cm. They both start up from the same point 0.
a) Find the point on the line closest to 0 where both fleas can land.
b) Describe all the spots on the line where both fleas can land.

Solutions:
a) The red flea lands at spots 10cm, 20cm, **30**cm, 40cm, 50cm, **60**cm,...
The blue flea lands at 6cm, 12cm, 18cm, 24cm, **30**cm, 36cm, 42cm, 48cm, 54cm, **60**cm..
The point on the line closest to 0 where both fleas can land is at 30cm.
b) Intuitively we can say that the pattern of jumps repeats every 30 steps, so all other spots are multiples of 30cm.



For a more rigorous proof, let us choose a random common landing spot and call it M.
Then M is a multiple of both 6 and 10:
$M = 6m = 10n$ for some natural numbers m and n. Divide by 2 to get
$3m = 5n$. Thus 3 divides $5n$. Since 3 is a prime and 3 doesn't divide 5, it follows that 3 divides n (the property of primes). Then $M = 10n$ is divisible by both 3 and 10, so by 30 (the divisor rule).

30 is called the least common multiple of 6 and 10.

The *least common multiple* of two numbers is the smallest number that both of them divide.
The least common multiple of a and b is usually denoted by $lcm(a, b)$.

Note that the lcm of two numbers can be smaller than their product.

Finding $lcm(a,b)$.

Example: We note that $6 = 2 \times 3$ and $10 = 5 \times 3$ and $30 = 2 \times 5 \times 3$.
Can we deduce a rule for calculating $lcm(a,b)$?

In general, we can write $a = \gcd(a,b) \times \frac{a}{\gcd(a,b)}$ and $b = \gcd(a,b) \times \frac{b}{\gcd(a,b)}$.
The natural numbers $\frac{a}{\gcd(a,b)}$ and $\frac{b}{\gcd(a,b)}$ have no common factors.
The example indicates taking
$lcm(a,b) = \gcd(a,b) \times \frac{a}{\gcd(a,b)} \times \frac{b}{\gcd(a,b)}$. After simplifying we get

The LCM Rule:

$$\boxed{lcm(a,b) = \frac{ab}{\gcd(a,b)}}$$

Proof of rule:
Step I: It is not difficult to argue that both a and b divide $\frac{ab}{\gcd(a,b)}$.
Indeed, $\frac{ab}{\gcd(a,b)} = a \times \frac{b}{\gcd(a,b)} = b \times \frac{a}{\gcd(a,b)}$.
Step II: Let's denote by M any random common multiple of a and b.
Prove that M is divisible by $\frac{ab}{\gcd(a,b)}$:
i) Let $M = a\,m = b\,n$ for some integers m and n. Divide by $\gcd(a,b)$ to get
ii) $\frac{a}{\gcd(a,b)} \times m = \frac{b}{\gcd(a,b)} \times n$.
iii) Hence $\frac{a}{\gcd(a,b)}$ divides $\frac{b}{\gcd(a,b)} \times n$, but $\frac{a}{\gcd(a,b)}$ is co-prime with $\frac{b}{\gcd(a,b)}$.
Hence $\frac{a}{\gcd(a,b)}$ divides n.
iv) Hence $M = b\,n$ is a multiple of both b and $\frac{a}{\gcd(a,b)}$. Since b and $\frac{a}{\gcd(a,b)}$ have no common factors, M is a multiple of their product $b \times \frac{a}{\gcd(a,b)}$ (by the divisor rule).

Example:
Find all possible pairs of two mystery numbers with the property that their gcd is 4 and their lcm is 360.

Solution: $lcm(a,b) = \gcd(a,b) \times \frac{a}{\gcd(a,b)} \times \frac{b}{\gcd(a,b)}$ so $360 = 4 \times \frac{a}{\gcd(a,b)} \times \frac{b}{\gcd(a,b)}$

$$90 = \frac{a}{\gcd(a,b)} \times \frac{b}{\gcd(a,b)}$$

As $\frac{a}{\gcd(a,b)}$ and $\frac{b}{\gcd(a,b)}$ have no common factor, we have the following cases:

i) $\frac{a}{\gcd(a,b)} = 1$ and $\frac{b}{\gcd(a,b)} = 90$. Then $a = 4$ and $b = 360$.

ii) $\frac{a}{\gcd(a,b)} = 2$ and $\frac{b}{\gcd(a,b)} = 45$. Then $a = 8$ and $b = 180$.
iii) $\frac{a}{\gcd(a,b)} = 5$ and $\frac{b}{\gcd(a,b)} = 18$. Then $a = 20$ and $b = 72$.
iv) $\frac{a}{\gcd(a,b)} = 9$ and $\frac{b}{\gcd(a,b)} = 10$. Then $a = 36$ and $b = 40$.

By comparing the prime factorisations of a and b we get:

Rule: $\mathrm{lcm}(a,b)$ = the product of all primes of a and b, each with its highest index between the prime factorizations of a and of b .

Practice exercises:
Find the least common multiple of a and b if
i) $a = 30$ and $b = 42$.
ii) $a = 2000$ and $b = 616$.

Activity Sheet

1. In each of the following cases, find $\gcd(a,b)$:

$a)\ \gcd(336, 54)$ $\quad b)\ \gcd(851, 966)$ $\quad c)\ \gcd(30^{10}, 333)$

2. Find the smallest number that can be written using only the symbols

$140, \quad 525, \quad +, \quad -$

and prove it.

3. Find gcd $(2^{17} + 2^5 - 1, 2^{12} - 2^5 - 1)$.

4. a) Prove that the following fractions are not in reduced form:

$$\frac{51}{21}, \quad \frac{501}{201}, \quad \frac{5001}{2001}, \quad \frac{50001}{20001},$$

b) Prove that for any natural number k, the fraction

$$\frac{1 + 5^{k+1} \times 2^k}{1 + 5^k \times 2^{k+1}}$$

is not in reduced form.

5. Find two mystery numbers knowing that if we divide one by 4 we get the same answer as when we divide the other one by 3, their gcd is 15 and their lcm is 180.