TWENTY NINTH IRISH MATHEMATICAL OLYMPIAD

Saturday, 23 April 2016

First Paper

Solutions

1. Proposed by Gordon Lessells.

   If the three-digit number ABC is divisible by 27, prove that the three-digit numbers BCA and CAB are also divisible by 27.

   Solution

   If 27 divides ABC then 3 divides $A+B+C$.

$$\begin{aligned} ABC - BCA &= 99A - 90B - 9C \\ &= 9(11A - 10B - C) \\ &= 9(11A + 11B + 11C - 21B - 12C) \\ &= 99(A + B + C) - 27(7B + 4C) \end{aligned}$$

   Therefore, if 27 divides ABC then 27 divides BCA.
   Now, applying the result just proved to BCA, we deduce that 27 also divides CAB.

2. Proposed by Jim Leahy.

In triangle ABC we have $|AB| \neq |AC|$. The bisectors of $\angle ABC$ and $\angle ACB$ meet AC and AB at E and F, respectively, and intersect at I. If $|EI| = |FI|$ find the measure of $\angle BAC$.

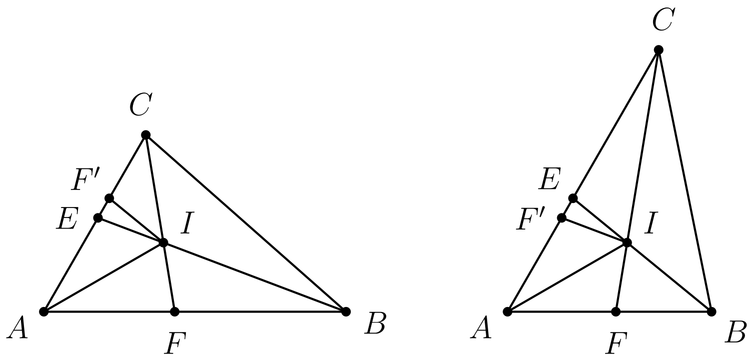
Solution

Let $\alpha = \angle BAC$, $\beta = \angle CBA$ and $\gamma = \angle ACB$, then $\alpha + \beta + \gamma = 180°$.

Step 1: Let F' be the reflection of F in the line AI.

Step 2: Because AI is the angle bisector of $\angle BAC$, F' is on the line AC. Moreover, because $|F'I| = |FI|$ and $|AF| = |AF'|$, the two triangles AFI and $AF'I$ are congruent. Note that $|F'I| = |EI|$ by assumption.

Step 3: We first show that $F' \neq E$. For a proof by contradiction we assume that $F' = E$. Then $\triangle AFI$ and $\triangle AEI$ are congruent, hence $\angle AFI = \angle AEI$. These two angles are external angles of the triangles BFC and BEC, respectively. Therefore, we obtain $\beta + \frac{1}{2}\gamma = \gamma + \frac{1}{2}\beta$, that is $\beta = \gamma$. This shows that $F' = E$ can only happen in a triangle with $|AB| = |AC|$, which was excluded.



Step 4: One of the two angles $\angle AF'I$ and $\angle AEI$ is a base angle of the isosceles triangle EIF' and the other of these two angles is an external angle at the base. Therefore, the sum of these two angles is equal to 180°.

Step 5: On the other hand, $\angle AF'I = \angle AFI$ is equal to $\beta + \frac{1}{2}\gamma$ (external angle of triangle BFC) and $\angle AEI = \gamma + \frac{1}{2}\beta$ (external angle of triangle BEC). Therefore, $180° = \frac{3}{2}(\beta + \gamma)$, and so $\beta + \gamma = 120°$. This implies that $\angle BAC = \alpha = 180° - 120° = 60°$.

3. Proposed by Mark Flanagan.

Do there exist four polynomials $P_1(x), P_2(x), P_3(x), P_4(x)$ with real coefficients, such that the sum of any three of them always has a real root, but the sum of any two of them has no real root?

Solution

There do not exist four such polynomials. We show this as follows.

Suppose that there do exist four such polynomials. If a polynomial has no real roots, it is either positive for all real x, or else it is negative for all real x. Consider the complete graph with the four polynomials as vertices. Colour the edge P_iP_j white if $P_i(x)+P_j(x) > 0$ for all real x, and black if $P_i(x)+P_j(x) < 0$ for all real x. We cannot have a triangle $P_iP_jP_k$ of the same colour (if we did, we could deduce that $P_i(x) + P_j(x) + P_k(x)$ has a constant sign for all real x, and therefore this sum of three polynomials would not have a real root). By the Pigeonhole Principle, at least three of the edges must have the same colour – let's say this colour is black. If they have a common vertex, then in order to avoid forming a black triangle, the other three edges would need to be white, thus forming a white triangle – a contradiction. So, without loss of generality, we can consider the case where P_1P_2, P_2P_3 and P_3P_4 are black. Then P_1P_3 and P_2P_4 must be white and therefore we have $(P_1(x) + P_3(x)) + (P_2(x) + P_4(x)) > 0$ for all real x. But since P_1P_2 and P_3P_4 must be black, we have $(P_1(x) + P_2(x)) + (P_3(x) + P_4(x)) < 0$ for all real x. The case where P_1P_2, P_2P_3 and P_3P_4 are white yields a similar contradiction. We conclude that there do not exist four such polynomials.

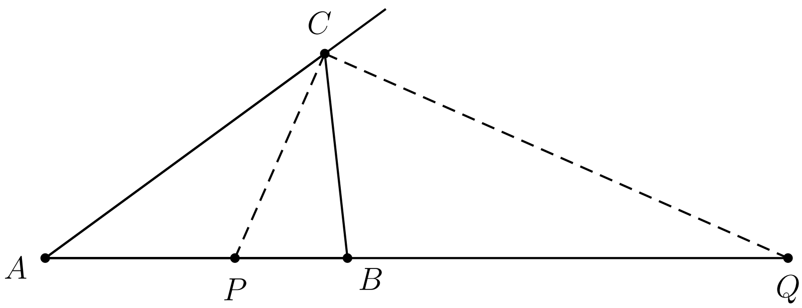
4. Proposed by Bernd Kreussler.

   Let ABC be a triangle with $|AC| \neq |BC|$. Let P and Q be the intersection points of the line AB with the internal and external angle bisectors at C, so that P is between A and B. Prove that if M is any point on the circle with diameter PQ, then $\angle AMP = \angle BMP$.

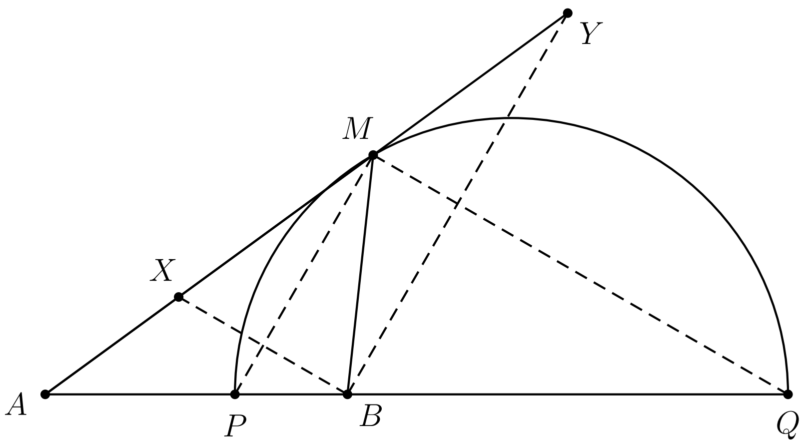
   Solution

   The internal and external angle bisectors divide the segment AB internally and externally in the same ratio $|AC| : |BC|$. This can be seen, for example, with the aid of the Sine Rule, applied to the triangles ACP and BCP for the internal bisector and to the triangles ACQ and BCQ for the external angle bisector. In particular, we obtain

$$\frac{|AP|}{|BP|} = \frac{|AQ|}{|BQ|}.$$



   Let X and Y be points on the line AM be such that $BX \| MQ$ and $BY \| MP$.



   Because $BX \| MQ$, we have $|AM|/|XM| = |AQ|/|BQ|$ and because $BY \| MP$, we have $|AM|/|YM| = |AP|/|BP|$. This implies $|AM|/|XM| = |AM|/|YM|$, hence $|XM| = |YM|$.

Because M is on the circle with diameter PQ, $\angle PMQ = 90°$ and so also $\angle YBX = 90°$. Therefore, B is on the circle with diameter XY. As M was shown to be the midpoint of XY, we obtain $MB = XM = YM$. This implies now

$$\frac{|AP|}{|BP|} = \frac{|AM|}{|BM|}.$$

If P' is the intersection point of the angle bisector of $\angle AMB$ and AB, then

$$\frac{|AP'|}{|BP'|} = \frac{|AM|}{|BM|} = \frac{|AP|}{|BP|}.$$

This implies that $P = P'$, and so MP is the angle bisector of $\angle AMB$.

Remark. The circle with diameter PQ is the circle of Apollonius which consists of all points M for which the ratio $|AM| : |BM|$ of the distances to the two points A and B is a given constant. In our case, this constant is equal to the ratio $|AC| : |BC|$. The two pairs of points A, B and P, Q form a so-called harmonic range.

5. Proposed by Gordon Lessells.

Let $a_1, a_2, \ldots, a_m$ be positive integers, none of which is equal to 10, such that $a_1 + a_2 + \cdots + a_m = 10m$. Prove that

$$(a_1 a_2 a_3 \cdots a_m)^{1/m} \leq 3\sqrt{11}.$$

Solution

Assume $m = 2k$ and that $a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_m$. Let

$$A_1 = \frac{a_1 + a_2 + \cdots + a_k}{k} \quad , \quad A_2 = \frac{a_{k+1} + a_{k+2} + \cdots + a_{2k}}{k} \; .$$

Clearly $A_1 \leq A_2$ and $\frac{A_1+A_2}{2} = 10$. Hence $A_1 = 10 - \delta$ and $A_2 = 10 + \delta$ for some $\delta \geq 0$.

If $\delta < 1$ then $A_1 > 9$ and $A_2 < 11$. The first of these inequalities implies that $a_k > 9$. This in turn implies that $a_k \geq 11$, which means that $a_{k+1}, a_{k+2}, \ldots, a_{2k} \geq 11$. But then we have $A_2 \geq 11$, which is a contradiction. Therefore we must have $\delta \geq 1$.

Now apply the AM-GM inequality separately to the numbers $a_1, a_2, \ldots, a_k$ and to the numbers $a_{k+1}, a_{k+2}, \ldots, a_{2k}$. We obtain

$$\prod_{i=1}^{m} a_i \leq (A_1 A_2)^k = (100 - \delta^2)^k \leq 99^k = 9^k 11^k$$

since $\delta \geq 1$.

Taking mth roots completes the proof for the case $m = 2k$.

To deal with the case where $m = 2k+1$, let $a_{m+1} = a_1,\ a_{m+2} = a_2,\ \ldots,\ a_{2m} = a_m$. Applying the result just proved to the numbers $a_1, a_2, \ldots, a_{2m}$ yields the result in this case.