TWENTY SEVENTH IRISH MATHEMATICAL OLYMPIAD

Saturday, 10 May 2014

First Paper

Solutions and Marking Schemes

1. Proposed by Steve Buckley.

   Given an 8×8 chess board, in how many ways can we select 56 squares on the board while satisfying both of the following requirements:

   (a) All black squares are selected.

   (b) Exactly seven squares are selected in each column and in each row.

   Solution

   Switching the definition of selection and non-selection, we see that an equivalent formulation is to select eight squares, with one in each row and in each column, and no black squares selected. We assume that the non-black squares are white.

   Orient the chessboard so that the square in row 1, column 1 is white. Let a_i be the column of the selected square in row i, so $i \mapsto a_i$ is a permutation of $\{1, \ldots, 8\}$ and a_i has the same parity as i. There are 4 choices for a_1 ($a_1 \in \{1, 3, 5, 7\}$), then 3 for a_3 ($a_3 \in \{1, 3, 5, 7\} \setminus \{a_1\}$), 2 for a_5, and 1 for a_7. After these selections, we again have 4 choices for a_2, 3 for a_4, etc. All of these sets of choices are independent of each other, so in all we have $(4!)^2 = 576$ ways of performing our selection.

   Proposed marking scheme

   - **2 marks** for noting the equivalent formulation in Paragraph 1.
   - **10 marks** for complete solution.

2. Proposed by Bernd Kreussler.

   Prove for all integers $N > 1$ that $(N^2)^{2014} - (N^{11})^{106}$ is divisible by $N^6 + N^3 + 1$.

   Solution:

   Observe that $2 \cdot 2014 - 11 \cdot 106 = 2862 = 9 \cdot 318$, hence

   $$\begin{aligned}(N^2)^{2014} - (N^{11})^{106} &= (N^{11})^{106} \left((N^9)^{318} - 1\right) \\ &= (N^{11})^{106}(N^9 - 1) \left((N^9)^{317} + (N^9)^{316} + \cdots + 1\right) .\end{aligned}$$

   Because $N^9 - 1 = (N^3 - 1)(N^6 + N^3 + 1)$, the above is divisible by $N^6 + N^3 + 1$.

Comment: As an alternative to direct factorization, consideration of the given number modulo $M = N^6 + N^3 + 1$ may be used to solve the problem. This type of solution uses the fact that $N^6 \equiv -(N^3+1) \bmod M$, so either of the observations $N^9 \equiv N^6 \cdot N^3 \equiv -(N^3+1)N^3 \equiv -(N^6+N^3) \equiv 1 \bmod M$ or $N^{12} \equiv (N^6)^2 \equiv (-N^3-1)^2 \equiv N^6 + 2N^3 + 1 \equiv N^3 \bmod M$ can be used to lead to a solution. For example, the first observation yields $N^{4028} - N^{1166} \equiv N^{9\cdot 447+5} - N^{9\cdot 129+5} \equiv N^5 - N^5 \equiv 0 \bmod M$.

Proposed marking scheme

- Up to **2 marks** can be given for showing this divisibility for a few specific values of N. But for **2 marks** it will be necessary to do substantial work beyond simple numerical calculations. These marks cannot be added to those mentioned below!
- A full solution gets **10 marks**. In case of an incomplete solution, partial marks could be awarded as follows:
- **2 marks** for producing a factor of the form $N^k - 1$;
- **3 marks** for observing that N^6+N^3+1 is a factor of N^9-1, or equivalent;
- **2 marks** for showing/observing that $N^{ak}-1$ is divisible by $N^a - 1$;
- **3 marks** for putting these together.
- For a solution which works modulo M, **3 marks** can be awarded for proving that $N^9 \equiv 1 \bmod M$. Having the idea to work modulo M, together with observation of the simple identity $N^6 \equiv -(N^3+1) \bmod M$, is worth a total of **2 marks**, which cannot be added to the aforementioned 3 marks.

3. Proposed by Jim Leahy.

In the triangle ABC, D is the foot of the altitude from A to BC, and M is the midpoint of the line segment BC. The three angles $\angle BAD$, $\angle DAM$ and $\angle MAC$ are all equal. Find the angles of the triangle ABC.

Solution 1

Step 1: Let $\angle BAD = \angle DAM = \angle MAC$. Since $\angle ADB = \angle ADM = 90°$, triangles ADB and ADM are similar.

Step 2: Therefore $BD = DM$, and so $DM = \frac{1}{2}BM = \frac{1}{2}MC$.

Step 3: Next, since AM bisects $\angle DAC$ we have $\frac{AC}{AD} = \frac{MC}{MD} = 2$.

Step 4: Therefore $\cos \angle DAC = 1/2$.

Step 5: So $\angle DAC = 60°$.

Step 6: It follows that $\angle ACD = 30°$ and $\angle DAM = 30°$.

Step 7: Therefore, $\angle BAC = 90°$ and $\angle ABC = 60°$. Therefore, the angles of the triangle ABC are $60°$, $30°$ and $90°$.

Solution 2

Step 1: Let $\angle BAD = \angle DAM = \angle MAC = \alpha$. Since $\angle ADB = \angle ADM = 90°$, triangles ADB and ADM are similar.

Step 2: Therefore $BD = DM = x$, and so $BM = MC = 2x$.

Step 3: Let $\tan\alpha = t$. Then, $\tan 2\alpha = 2t/(1-t^2)$. So, letting $AD = h$, we have

$$\frac{3x}{h} = \frac{2\left(\frac{x}{h}\right)}{1-\left(\frac{x}{h}\right)^2} = \frac{2xh}{h^2 - x^2}$$

Step 4: Rearranging gives $\frac{x}{h} = \tan\alpha = \frac{1}{\sqrt{3}}$, and so $\alpha = 30°$.

Step 5: So $\angle DAC = 2\alpha = 60°$ and so $\angle ACD = 30°$.

Step 6: Then, $\angle BAC = 3\alpha = 90°$, and so $\angle ABC = 60°$. Therefore, the angles of the triangle ABC are $60°$, $30°$ and $90°$.

Proposed Marking Scheme:

Solution 1:

- **2 marks** for *Step 1*.
- **2 mark** for *Step 2*.
- **2 marks** for *Step 3*.
- **1 mark** for *Step 4*.
- **1 mark** for *Step 5*.
- **1 mark** for *Step 6*.
- **1 mark** for *Step 7*.

Solution 2:

- **2 marks** for *Step 1*.
- **2 mark** for *Step 2*.
- **2 marks** for *Step 3*.
- **2 mark** for *Step 4*.
- **1 mark** for *Step 5*.
- **1 mark** for *Step 6*.

4. Proposed by Mark Flanagan.

Three different nonzero real numbers a, b, c satisfy the equations

$$a+\frac{2}{b}=b+\frac{2}{c}=c+\frac{2}{a}=p$$

where p is a real number. Prove that $abc+2p=0$.

Solution 1:

We eliminate variables to get an equation in b and p. First,

$$a=p-\frac{2}{b}=\frac{bp-2}{b}, \text{ so that } \frac{1}{a}=\frac{b}{bp-2}\ .$$

Then,

$$c=p-\frac{2}{a}=p-\frac{2b}{bp-2}=\frac{bp^2-2p-2b}{bp-2}, \text{ so that } \frac{1}{c}=\frac{bp-2}{bp^2-2p-2b}\ .$$

Finally,

$$b=p-\frac{2}{c}=p-\frac{2bp-4}{bp^2-2p-2b}\ .$$

Rearranging this equation we obtain

$$b^2(p^2-2)+b(2p-p^3)+(2p^2-4)=0\ .$$

If $p^2\neq 0$, b must be a root of the quadratic function $f(x)=x^2(p^2-2)+x(2p-p^3)+(2p^2-4)$ and by symmetry so must a and c. But this is a contradiction since a, b and c are different numbers. It follows that we must have $p^2=2$ and so p must be equal either $+\sqrt{2}$ or $-\sqrt{2}$.

It is easy to check that both possibilities for p lead to the solution set

$$a=\frac{pt-2}{t}\ ;\ b=t\ ;\ c=-\frac{2p}{pt-2}\ .$$

where $t\neq 0, \frac{2}{p}$. This solution must satisfy $abc=-2p$.

Solution 2:

The equation $a+\frac{2}{b}=b+\frac{2}{c}$ implies $a-b=\frac{2}{c}-\frac{2}{b}$ which can be rewritten as

$$bc(a-b)=2(b-c)\ . \qquad (1)$$

Because the given equations are cyclically symmetric, we also obtain

$$ab(c-a)=2(a-b) \quad \text{and} \quad ac(b-c)=2(c-a)\ .$$

Multiplying these three equations and using that $(a-b)(b-c)(c-a)\neq 0$, we obtain

$$(abc)^2=8\ .$$

This implies that $abc = \varepsilon 2\sqrt{2}$ with $\varepsilon \in \{1, -1\}$ simply a sign.

If we now multiply the original equations with $abc = \varepsilon 2\sqrt{2}$ we obtain, after cancelling the common factor 2,

$$a(c + \varepsilon\sqrt{2}) = b(a + \varepsilon\sqrt{2}) = c(b + \varepsilon\sqrt{2}) = \varepsilon\sqrt{2}p\,.$$

Using one of these identities, e.g. $b(a + \varepsilon\sqrt{2}) = \varepsilon\sqrt{2}p$, together with $p = a + \dfrac{2}{b}$ we get $b^2(a + \varepsilon\sqrt{2}) = \varepsilon\sqrt{2}(ab + 2)$. This equation can be rearranged into

$$ab\left(b - \varepsilon\sqrt{2}\right) = -\varepsilon\sqrt{2}\left(b^2 - 2\right) = -\varepsilon\sqrt{2}\left(b + \varepsilon\sqrt{2}\right)\left(b - \varepsilon\sqrt{2}\right)\,.$$

If $b \neq \varepsilon\sqrt{2}$ we obtain $a = -\frac{\varepsilon\sqrt{2}\left(b+\varepsilon\sqrt{2}\right)}{b} = -\varepsilon\sqrt{2} - \frac{2}{b}$ and hence

$$p = a + \frac{2}{b} = -\varepsilon\sqrt{2}\,.$$

Using cyclic symmetry again, we similarly obtain

$$bc\left(c - \varepsilon\sqrt{2}\right) = -\varepsilon\sqrt{2}\left(c^2 - 2\right) = -\varepsilon\sqrt{2}\left(c + \varepsilon\sqrt{2}\right)\left(c - \varepsilon\sqrt{2}\right)\,.$$

If $c \neq \varepsilon\sqrt{2}$, this equation implies as above $p = -\varepsilon\sqrt{2}$.

By assumption $b \neq c$, hence $b \neq \varepsilon\sqrt{2}$ or $c \neq \varepsilon\sqrt{2}$ and either way we obtain $p = -\varepsilon\sqrt{2}$. This shows that the possible values for p are $\pm\sqrt{2}$ and also that $abc + 2p = 0$.

Finally, with $a = 1$, $b = 2 - 2\varepsilon\sqrt{2}$ and $c = -2 - \varepsilon\sqrt{2}$ we easily indeed obtain $p = -\varepsilon\sqrt{2}$.

Proposed Marking Scheme:

Solution 1:

- **3 marks** for eliminating all but one of the variables a, b, c and obtaining a quadratic in one variable. **Up to 2 marks** can be awarded for a convincing but unsuccessful effort to make appropriate substitutions so as to eliminate all but one of the variables a, b, c.
- **2 marks** for observing that a, b and c must *all* be roots of the quadratic.
- **1 mark** for noting that this leads to a contradiction.
- **1 mark** for deducing $p^2 = 2$.
- **2 mark** for writing down the one-dimensional solution set.
- **1 mark** for showing that this satisfies $abc + 2p = 0$.

Solution 2:

- **2 marks** for rearranging any of the given equations into the form (1).
- **1 mark** for writing down the other two equations in the same form.
- **3 marks** for multiplying the three equations together and cancelling to get $(abc)^2 = 8$.
- **4 marks** for correctly finishing the solution.

5. Proposed by Steve Buckley.

Suppose $a_1, \dots, a_n > 0$, where $n > 1$ and $\sum_{i=1}^n a_i = 1$. For each $i = 1, 2, \dots, n$, let $b_i = a_i^2 / \sum_{j=1}^n a_j^2$. Prove that

$$\sum_{i=1}^n \frac{a_i}{1 - a_i} \le \sum_{i=1}^n \frac{b_i}{1 - b_i} \,.$$

When does equality occur?

Solution

Without loss of generality, we assume that $a_1 \ge a_2 \ge \cdots \ge a_n$, so also $b_1 \ge b_2 \ge \cdots \ge b_n$ and $0 < a_i, b_i < 1$. We write $A_k = \sum_{i=1}^k a_i$ and $B_k = \sum_{i=1}^k b_i$ for $0 \le k \le n$.

We first claim that $B_k \ge A_k$ for all $0 \le k \le n$. Since $B_0 = A_0 = 0$ and $B_n = A_n = 1$, we may assume that $0 < k < n$. Writing

$$S = \sum_{i=1}^n a_i^2 = \sum_{i=1}^n a_i^2 \cdot \sum_{i=1}^n a_i \,,$$

we see that

$$\begin{aligned}(B_k - A_k)S &= \sum_{i=1}^k \sum_{j=1}^n a_i^2 a_j - \sum_{i=1}^k \sum_{j=1}^n a_i a_j^2 \\ &= \sum_{\substack{1 \le i \le k \\ k < j \le n}} a_i a_j (a_i - a_j) \ge 0 \,,\end{aligned}$$

because all terms in the sum are non-negative. But S is positive, so $B_k \ge A_k$, as claimed. Note also that one of the terms in the sum, namely $a_1 a_n (a_1 - a_n)$, is positive unless $a_1 = \cdots = a_n$, so we have $B_k = A_k$ for $0 < k < n$ in this case, and $B_k > A_k$ for $0 < k < n$ otherwise.

Next, let $f(x) = x/(1 - x)$ and let $g(x, y) = 1/(1 - y)(1 - x)$. Note that $g(x, y) > 0$ for all $0 < x, y < 1$, and that g is increasing as a function of both x and y. Moreover, if $0 < x, y < 1$ and $x \ne y$, then

$$\frac{f(y) - f(x)}{y - x} = g(x, y) > 0 \,. \qquad (2)$$

We wish to prove that $F := \sum_{i=1}^{n}(f(b_i) - f(a_i)) \geq 0$. Writing $d_i = b_i - a_i$, $D_i := B_i - A_i$, $a_{n+1} = b_{n+1} = 0$, and $c_i = g(b_i, a_i)$ for $1 \leq i \leq n+1$, it follows from (2) that

$$\begin{aligned} F := \sum_{i=1}^{n}(f(b_i) - f(a_i)) &= \sum_{i=1}^{n} c_i d_i \\ &= \sum_{i=1}^{n} c_i(D_i - D_{i-1}) \\ &= c_n D_n + \sum_{i=1}^{n}(c_i - c_{i+1})D_i\,. \end{aligned}$$

Now $D_n = 0$. As for the sum in the last expression, it is non-negative because both factors are non-negative for all i (the first one because g is increasing as a function of both of its arguments). Thus $D \geq 0$, as required.

As for equality, we have $D = 0$ if and only if every term in the last sum equals 0. If $a_1 = \cdots = a_n$, then each $D_i = 0$ and the sum is zero. If however not all numbers a_i are equal, then we have seen that each D_i, $1 \leq i < n$, is positive. In particular, $D_1 > 0$, and so $b_1 > a_1$ and $c_1 > 0$. Since $c_{n+1} = 0$, we must have $c_j > c_{j+1}$ for some $1 \leq j \leq n$, and the term $(c_j - c_{j+1})D_j$ in our last sum is positive, ensuring that $F > 0$. Thus the condition for equality is that all the numbers a_i are equal.

Proposed marking scheme:

- **3 marks** for a proof that $B_k \geq A_k$, or **1 mark** if this is claimed without proof.
- **1 mark** for noting equation (1); we do not insist on the obvious restriction $x \neq y$.
- A total of **8 marks** for a complete proof of the inequality, with the remaining **2 marks** for a complete proof of the condition for equality.