TWENTY SIXTH IRISH MATHEMATICAL OLYMPIAD

Saturday, 11 May 2013

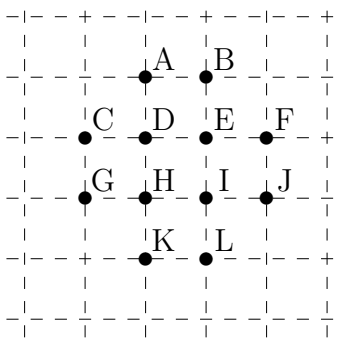
Second Paper

Time allowed: **Three hours.**

6. The three distinct points B, C, D are collinear with C between B and D. Another point A not on the line BD is such that $|AB| = |AC| = |CD|$.

   Prove that $\angle BAC = 36°$ if and only if $\dfrac{1}{|CD|} - \dfrac{1}{|BD|} = \dfrac{1}{|CD| + |BD|}$.

7. Consider the collection of different squares which may be formed by sets of four points chosen from the 12 labelled points in the diagram on the right.

   For each possible area such a square may have, determine the number of squares which have this area.

   Make sure to explain why your list is complete.

8. Find the smallest positive integer N for which the equation

   $$(x^2 - 1)(y^2 - 1) = N$$

   is satisfied by at least two pairs of integers (x, y) with $1 < x \leq y$.

9. We say that a doubly infinite sequence

   $$\ldots, s_{-2}, s_{-1}, s_0, s_1, s_2, \ldots$$

   is *subaveraging* if $s_n = (s_{n-1} + s_{n+1})/4$ for all integers n.

   (a) Find a subaveraging sequence in which all entries are different from each other. Prove that all entries are indeed distinct.

   (b) Show that if (s_n) is a subaveraging sequence such that there exist distinct integers m, n such that $s_m = s_n$, then there are infinitely many pairs of distinct integers i, j with $s_i = s_j$.

10. Let a, b, c be real numbers and let

    $$x = a + b + c,\ y = a^2 + b^2 + c^2,\ z = a^3 + b^3 + c^3 \quad \text{and} \quad S = 2x^3 - 9xy + 9z.$$

    (a) Prove that S is unchanged when a, b, c are replaced by $a + t, b + t, c + t$, respectively, for any real number t.

    (b) Prove that $\left(3y - x^2\right)^3 \geq 2S^2$.