TWENTY SIXTH IRISH MATHEMATICAL OLYMPIAD

Saturday, 11 May 2013

First Paper

Time allowed: **Three hours.**

1. Find the smallest positive integer m such that $5m$ is an exact 5th power, $6m$ is an exact 6th power, and $7m$ is an exact 7th power.

2. The altitudes of a triangle ABC are used to form the sides of a second triangle $A_1B_1C_1$. The altitudes of $\triangle A_1B_1C_1$ are then used to form the sides of a third triangle $A_2B_2C_2$. Prove that $\triangle A_2B_2C_2$ is similar to $\triangle ABC$.

3. Prove that

$$1 - \frac{1}{2012}\left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2013}\right) > \frac{1}{\sqrt[2012]{2013}}.$$

4. A *Colombian Square* is a 6×6 square which is subdivided into 36 unit squares, each of which is coloured either Yellow, Blue or Red according to the following rules:

   (a) No row or column may contain more than two unit squares of the same colour.

   (b) In any set of four unit squares obtained by intersecting two rows with two columns, no colour is to occur exactly three times.

   How many different Colombian Squares are there?

5. A, B and C are points on the circumference of a circle with centre O, such that $\triangle ABC$ is not a right-angled triangle. The point P lies on the circumcircle Γ_1 of the triangle OAB such that OP is a diameter of Γ_1. The point Q lies on the circumcircle Γ_2 of the triangle OAC such that OQ is a diameter of Γ_2. Tangents are drawn to the circles Γ_1 and Γ_2 at P and Q respectively; these two tangents intersect at K. The line CA meets the circle Γ_1 at A and X. Prove that X lies on the line KO.