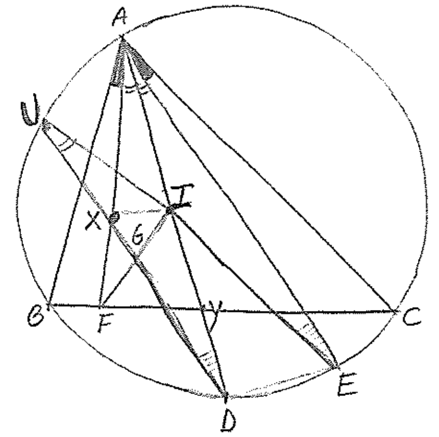
IMO 2010 Problem 2



We know:

- AD angle bisector of $\widehat{BAC}$
- $I =$ incentre of ΔABC ($\Leftrightarrow$ BI, CI angle bisectors)
- $\widehat{BAF} = \widehat{CAE}$
- $G =$ midpoint of FI
- DG intersects EI at U.

To prove:

U is on the circumcircle of ΔABC.

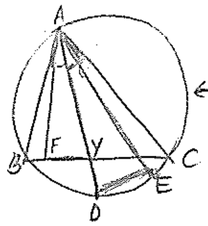
Strategy $\Rightarrow$ Cyclic quadrilateral: We should prove UAED cyclic.
Using the angles already available, to prove: either
$\widehat{DUE} = \widehat{DAE}$ or $\widehat{UDA} = \widehat{UEA}$.
The second pair looks more promising, because we know more about the vertices D and E than about U, which was constructed indirectly from D and E.

The data in the statement of the problem are a mixture of angles and segment equalities, so our strategy should mix angles & segment lengths. Most likely, *similar triangles*. (Also keep an eye open for other methods using angle bisectors, midpoints).

Hint ① Try finding a pair of Δ-s which should be similar and include the angles $\hat{D}$ and $\hat{E}$. Again, avoid the obvious ($\Delta UID \sim \Delta AIE$) since we know too little about U, |UI|, |UD|.
Let AF intersect DG at X.

Hint ② If you've decided to prove $\Delta AXD \sim \Delta AIE$, can you find a related pair of similar triangles?
(Hint: ΔAXD and ΔAIE are "rotated" triangles - see basic toolkit).

Hint ③



Forget about U for a minute
$\leftarrow$ Which pairs of similar triangles can you find in this simplified diagram?
One of them should include ΔADE.

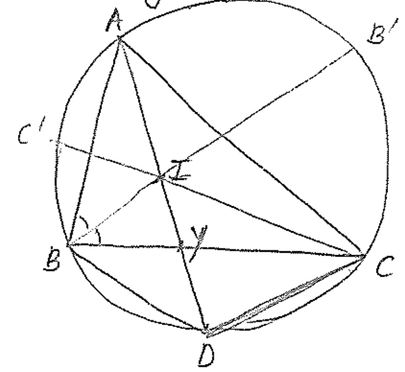
Hint ④ To recap: $\triangle AFY \sim \triangle ADE$. To prove: $\triangle AXI \sim \triangle ADE$.

It is sufficient to prove $XI \parallel FY$.

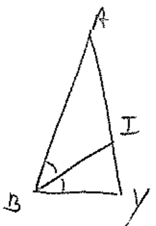
Again, try to use ratios: $\dfrac{|AX|}{|XF|} = \dfrac{|AI|}{|IY|}$.

Use G = midpoint of FI and Menelaus' theorem to write $\dfrac{|AX|}{|XF|}$ in terms of segments on the line AD.

Hint ⑤ The problem is reduced to a property of the diagram: I = incentre, AD = ∡ bisector of $\hat{A}$



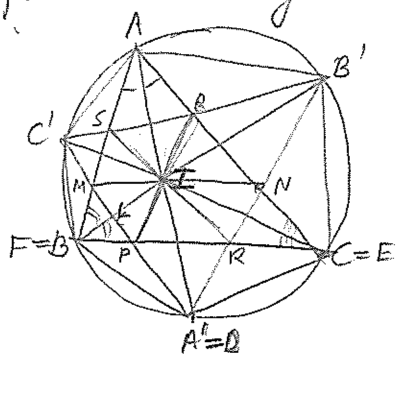
- Prove: • $|DB| = |DI| = |DC|$.
  - $\triangle ABY \sim \triangle ADC$.
- Use the property of angle bisectors which involves ratios of segments for



to reduce $\dfrac{|AI|}{|IY|}$ to $\dfrac{|AB|}{|BY|}$

<u>Conclusion</u>: In retrospect, our strategy has been to reduce the initial complicated diagram to the more basic ones from Hint ③ and then Hint ⑤, by gradually eliminating elements which are extrinsic to $\triangle ABC$, and reducing to a problem about $\triangle ABC$ and its intrinsic elements (incentre etc)

Alternately, one could start by solving an extrinsic problem for $\triangle ABC$ by considering the special case when $B=F$ and $C=E$:

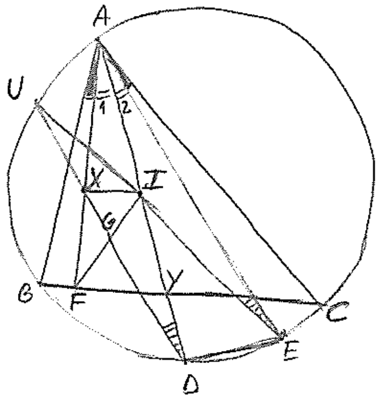


I = incentre of $\triangle ABC$. What special properties can you discover here? E.g., can you prove:

- $|A'B| = |A'C| = |A'I|$, $|C'A| = |C'B| = |C'I|$, etc.
- I = orthocentre of $\triangle A'B'C'$ and L = midpoint of BI
- $MIPB$, $NIRC$, $SIQA$ rhombuses.
- M, I, N collinear, and $MN \parallel BC$...

IMO 2010 Problem 2

Solution 1



U = intersection of DX and IE

U on circumcircle of $\triangle ABC \iff AUDE$ cyclic

$\iff \widehat{UDA} = \widehat{UEA} \iff \triangle AXD \sim \triangle AIE$ (as $\widehat{A}_1 = \widehat{A}_2$)

"Rotated $\triangle$-s" $\iff \triangle AXI \sim \triangle ADE$.

But $\triangle AFY \sim \triangle ADE$

(because $\widehat{A}_1 = \widehat{A}_2$ and $\widehat{AYF} = \dfrac{\overset{\frown}{AB} + \overset{\frown}{CD}}{2} = \dfrac{\overset{\frown}{AB} + \overset{\frown}{BD}}{2} = \dfrac{\overset{\frown}{AD}}{2} = \widehat{AED}$)

Thus remains to prove:

$\triangle AXI \sim \triangle AFY$

$\iff XI \parallel FY \iff \boxed{\dfrac{|AX|}{|XF|} = \dfrac{|AI|}{|IY|}}$ $\Rightarrow$

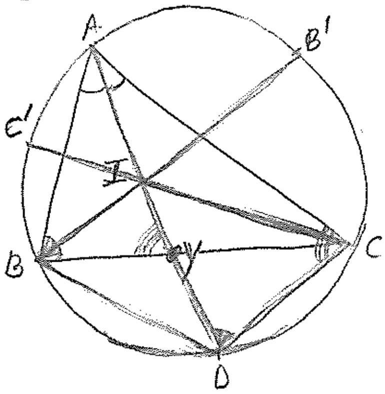
G = midpoint of FI.

Menelaus' theorem in $\triangle AFI$ crossed by $XGD \Rightarrow \dfrac{|AX|}{|XF|} \cdot \dfrac{|FG|}{|GI|} \cdot \dfrac{|ID|}{|DA|} = 1$

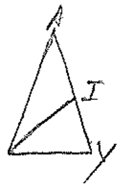
$\Rightarrow \boxed{\dfrac{|AX|}{|XF|} = \dfrac{|DA|}{|DI|}}$

Remains to prove: $\boxed{\dfrac{|AI|}{|IY|} = \dfrac{|AD|}{|DI|}}$

We reduced the problem to a property of the <u>angle bisector</u> <u>AD</u> and <u>incentre I</u>:



- BI angle bisector in $\triangle ABY$



$\Rightarrow \boxed{\dfrac{|AI|}{|IY|} = \dfrac{|AB|}{|BY|}} \left(= \dfrac{Area(ABI)}{Area(YBI)}\right)$

- $\triangle ABY \sim \triangle ADC$ because $\widehat{BAD} = \widehat{DAC}$ and $\widehat{ABY} = \widehat{ADC} = \dfrac{\overset{\frown}{AC}}{2}$ $\Rightarrow \boxed{\dfrac{|AB|}{|BY|} = \dfrac{|AD|}{|DC|}}$

- $\boxed{|DC| = |DI| = |DB|}$ because

$\triangle BDI$ and $\triangle IDC$ isosceles because

$\widehat{IBD} = \dfrac{\overset{\frown}{DB'}}{2} = \dfrac{\overset{\frown}{DC} + \overset{\frown}{CB'}}{2} = \dfrac{\widehat{A}}{2} + \dfrac{\widehat{B}}{2}$

$\widehat{BID} = \dfrac{\overset{\frown}{BD} + \overset{\frown}{AB'}}{2} = \dfrac{\widehat{A}}{2} + \dfrac{\widehat{B}}{2}$

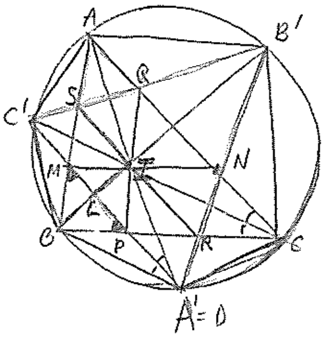
$\Rightarrow \widehat{IBD} = \widehat{BID} \Rightarrow \triangle BDI$ isosceles

similarly $\triangle CDI$ isosceles

$\Rightarrow$ in conclusion $\boxed{\dfrac{|AI|}{|IY|} = \dfrac{|AD|}{|DI|}}$.

IMO 2010 Problem 2

Solution 2 Part I The case when $F = B$ and $E = C$:



AA', BB', CC' angle bisectors of $\hat{A}, \hat{B}, \hat{C}$ respectively.

$\Rightarrow \overset{\frown}{BA'} = \overset{\frown}{A'C} = \hat{A}$, $\overset{\frown}{BC'} = \overset{\frown}{C'A} = \hat{C}$, $\overset{\frown}{AB'} = \overset{\frown}{B'C} = \hat{B}$.

$\Rightarrow \widehat{B'BA'} = \dfrac{\overset{\frown}{A'B'}}{2} = \dfrac{\overset{\frown}{A'C} + \overset{\frown}{CB'}}{2} = \dfrac{\hat{A}+\hat{B}}{2}$ $\Big\}$ $\Rightarrow \widehat{B'BA} = \widehat{BIA'}$

$\widehat{BIA'} = \dfrac{\overset{\frown}{BA'} + \overset{\frown}{AB'}}{2} = \dfrac{\hat{A}+\hat{B}}{2}$ $\Rightarrow \Delta IA'B$ isosceles

and $\widehat{BA'C'} = \dfrac{\overset{\frown}{BC'}}{2} = \dfrac{\overset{\frown}{C'A}}{2} = \widehat{C'A'A}$ $\Big\} \Rightarrow$

$\Rightarrow$ $A'C'$ = angle bisector in $\Delta IA'B$ isosceles with $|A'I| = |A'B|$

$\Rightarrow \boxed{A'C' = \perp \text{ bisector of } BI}$ and P, M on $A'C'$ $\Rightarrow \boxed{|PB| = |PI|}$, $\boxed{MI = MB}$ $\Big\} \Rightarrow$

Also, $\widehat{BMP} = \dfrac{\overset{\frown}{AC'} + \overset{\frown}{BA'}}{2} = \dfrac{\hat{C}+\hat{A}}{2}$

$\widehat{BPM} = \dfrac{\overset{\frown}{A'C} + \overset{\frown}{BC'}}{2} = \dfrac{\hat{A}+\hat{C}}{2}$ $\Big\} \Rightarrow \widehat{BMP} = \widehat{BPM}$

ΔBMP isosceles with $\boxed{|BM| = |MP|}$ and BB' = angle bisector $\Rightarrow \perp$ bisector of MP

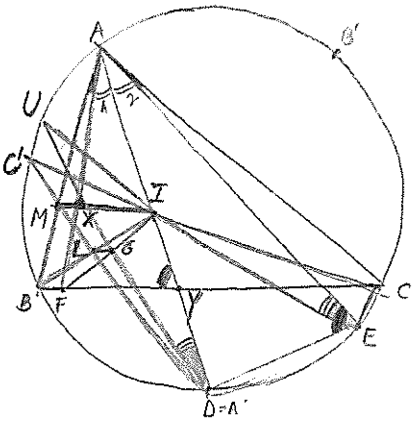
$\Rightarrow$ $BMIP$ rhombus, thus its diagonals $BI \perp MP$ and and they intersect at their midpoints.

$PI \parallel BM$, $\boxed{MI \parallel BP}$,

Similarly, $\boxed{IN \parallel CR}$ $\Big\} \Rightarrow \boxed{M, I, N \text{ collinear and } MN \parallel BC}$

We also proved $\boxed{L = \text{midpoint of } BI}$ and $\boxed{AL, CI \text{ intersect at } C' \text{ on } \mathcal{C}(ABC)}$.

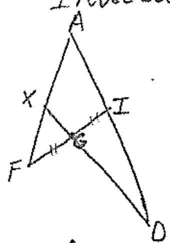
Part II Compare to general case:



Here G = midpoint of IF, L = midpoint of BI, UD intersects AF at X and AB intersects DC' at M.

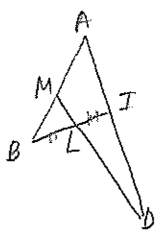
We note that $\boxed{M, X \text{ and } I \text{ are collinear}}$.

Indeed, we use G and L:



Menelaus' theorem for ΔAFI cut by XGD:

$\dfrac{|AX|}{|XF|} \cdot \dfrac{\cancel{|FG|}}{\cancel{|GI|}} \cdot \dfrac{|ID|}{|DA|} = 1 \Rightarrow \dfrac{|AX|}{|XF|} = \dfrac{|DA|}{|ID|}$ $\Big\} \Rightarrow$



similarly, Menelaus' theorem for ΔABI cut by MLD:

$\Rightarrow \dfrac{|AM|}{|MB|} = \dfrac{|DA|}{|ID|}$

$\Rightarrow \dfrac{|AX|}{|XF|} = \dfrac{|AM|}{|MB|} \Rightarrow \boxed{MX \parallel BC}$. But $\boxed{MI \parallel BC}$ $\Rightarrow M, X, I$ collinear

Part III Use $XI \parallel BC$ to solve the general case:

$\triangle AXI \sim \triangle AFY$

But $\triangle AFY \sim \triangle ADE$ $\left(\text{because } \widehat{FAY} = \widehat{DAE} \text{ and } \widehat{AYF} = \dfrac{\overset{\frown}{AB} + \overset{\frown}{DC}}{2} = \dfrac{\overset{\frown}{AB} + \overset{\frown}{DB}}{2} = \dfrac{\overset{\frown}{AD}}{2} = \widehat{AEB}\right)$

$\Rightarrow \triangle AXI \sim \triangle ADE$ "rotated $\triangle$-s"

$\Rightarrow \boxed{\triangle AXD \sim \triangle AIE}$

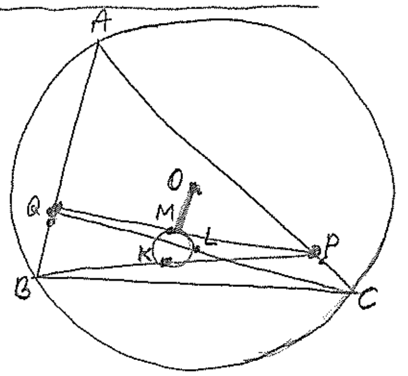
(because $\hat{A}_1 = \hat{A}_2$ and $\dfrac{|AX|}{|AI|} = \dfrac{|AD|}{|AE|}$)

$\triangle AXD \sim \triangle AIE \Rightarrow \widehat{ADX} = \widehat{AEI}$

$\Rightarrow AEDU$ cyclic

$\Rightarrow U$ on the circumcircle of $\triangle ABC$.

IMO 2009 Problem 2



We know:

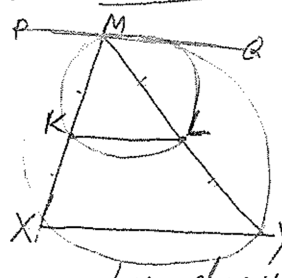
- O = circumcentre of $\triangle ABC$. P, Q points on AC, AB
- K = midpoint of BP
- L = midpoint of CQ
- M = midpoint of QP.
- Circumcircle of $\triangle MKL$ ($\mathcal{C}(MKL)$) is tangent to PQ

To prove: $|OP| = |OQ|$.

Strategy → Note: since M = midpoint of QP, if $|OP| = |OQ|$ then $OM \perp PQ$ and since $\mathcal{C}(MKL)$ is tangent to PQ, then its circumcentre S would lie on the line OM. In fact, the reciprocal is also true, since $SM \perp PQ$, so if O is on the line SM, then $OM \perp PQ \Rightarrow OM = \perp$ bisector of PQ.
(M = midpoint of PQ)

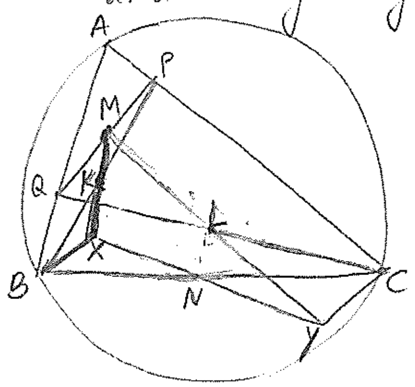
→ The diagram is too small! The midpoints M, K, L are too close together. To make the diagram larger, extend ML and MK to double their sizes. This technique is very appropriate here since MK, ML are midlines, so $|MK| = \frac{|BQ|}{2}$, $|ML| = \frac{|PC|}{2}$;

Also, use Lemma:



$$\boxed{\begin{array}{l} KL \parallel XY \\ \Rightarrow \mathcal{C}(MKL) \text{ is tangent to } \\ \quad \mathcal{C}(MXY) \end{array}}$$

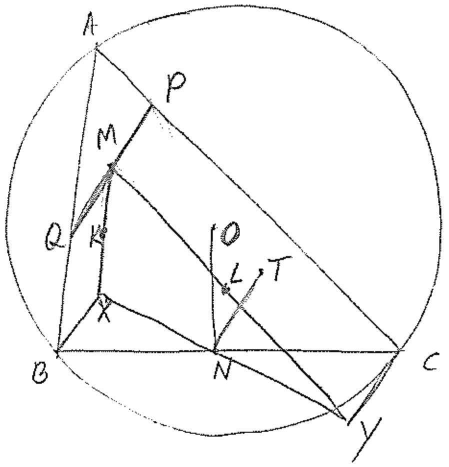
Hint ① Prove Lemma by drawing the common tangent PQ and using angles and arcs arguments.



Hint ② Parallel lines, equal segments lead to a number of parallelograms and further equal segments.
If K = midpoint of MX, L = midpoint of MY, can you prove that N = midpoint of both BC and XY?

(This question is very similar with one in ExSet3)

IMO 2009 Problem 2 Solution:



Let $|MX| = 2|MK|$ and $|MY| = 2|ML|$

As MK = midline in $\triangle PQB$

$\Rightarrow |MK| = \frac{1}{2}|BQ|$ and $MK \parallel BQ$

$\Rightarrow |MX| = |BQ|$, $MX \parallel BQ$

$\Rightarrow MXBQ$ parallelogram

$\Rightarrow MQ \parallel BX$ and $|MQ| = |BX|$.

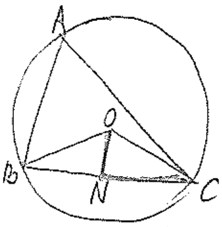
Similarly $MP \parallel CY$ and $|MP| = |CY|$.

Also, we know $|MQ| = |MP|$ $\Rightarrow$

$\Rightarrow |BX| = |CY|$. As $BX \parallel CY$ as well, $\Rightarrow BXCY$ parallelogram

$\Rightarrow$ diagonals BC and XY intersect at midpoint N.

(see Lecture notes for details on parallelograms).



O = circumcentre of $\triangle ABC$

$\Rightarrow \widehat{BOC} = 2\widehat{NOC} = 2\hat{A} \Rightarrow \widehat{NOC} = \hat{A}$

But $MX \parallel BQ$ and $MY \parallel PC \Rightarrow \hat{A} = \hat{M}$

similarly, T = circumcentre of $\triangle MXY \Rightarrow \widehat{NTY} = \hat{M}$

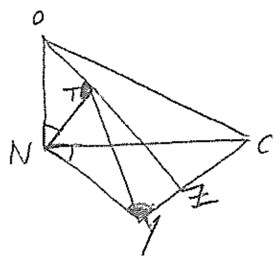
$\Rightarrow \widehat{NOC} = \widehat{MTY}$.

Also, $\widehat{ONC} = \widehat{TNY}$ $\Rightarrow$

$\Rightarrow \boxed{\triangle ONC \sim \triangle TNY}$

"Rotated $\triangle$-s"

$\Rightarrow \triangle ONT \sim \triangle CNY$ $\left(\text{as } \widehat{ONT} = \widehat{CNY} \text{ and } \dfrac{|ON|}{|CN|} = \dfrac{|NT|}{|NY|}\right)$



Let OT intersect CY at Z.

As $\widehat{OTN} = \widehat{NYC} \Rightarrow TNYZ$ cyclic quadrilateral

$\Rightarrow \widehat{TNY} + \widehat{TZY} = 180°$

But $\widehat{TNY} = 90°$

$\Rightarrow \widehat{TZY} = 90°$

$\Rightarrow OT \perp CY$

As $CY \parallel PQ$

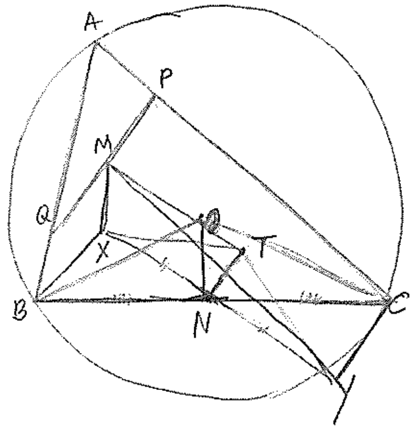
$\Rightarrow OT \perp PQ$ $\Rightarrow$

$\triangle MKL$ tangent to PQ, $KL \parallel XY$ $\Rightarrow \triangle MXY$ tangent to PQ $\Longrightarrow$ ~~But~~ $TM \perp PQ$

$\Rightarrow OM \perp PQ$

M = midpoint of PQ $\Rightarrow$ $\boxed{OM = \perp \text{ bisector of } PQ}$

$\Rightarrow \boxed{|OP| = |OQ|}$



Hint ③ Let $T =$ circumcentre of $\triangle MXY$.

To prove: T, O, M collinear;

Relevant data we know: $\boxed{TM \perp PQ}$

(because $\mathcal{C}(MXY)$ tangent to PQ)

Also, $PQ \parallel BX$ and $\boxed{PQ \parallel CY}$ $\Big\} \Rightarrow$

$\Rightarrow$ to prove: $OT \perp CY$.

Hint ④ Prove $\triangle ONC \sim \triangle TNY$ and then use the "rotated $\triangle$-s" Lemma.

Hint ⑤ Write $\widehat{NOC}$ in terms of $\hat{A}$ and $\widehat{NTY}$ in terms of $\hat{M}$.

What's the relation between $\hat{A}$ and $\hat{M}$?

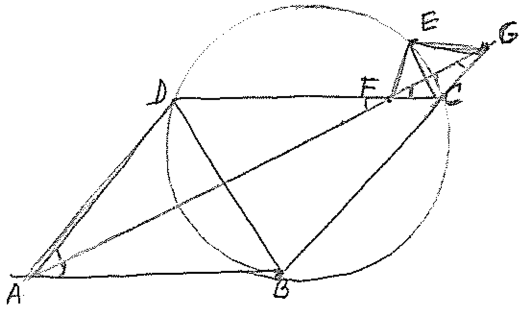
The Lemma about $\triangle MKL$ and $\triangle MXY$ is an exercise in Exercise Set 3 in the Geometry website accesible from

http://euclid.ucc.ie/pages/MATHENR/index.htm

Mathematics Notes

Geometry section.

IMO 2007 Problem 2



We know:

- $ABCD$ parallelogram
- $BCED$ cyclic
- $|EF| = |EC| = |EG|$

To prove: AF = angle bisector of $\widehat{BAD}$.

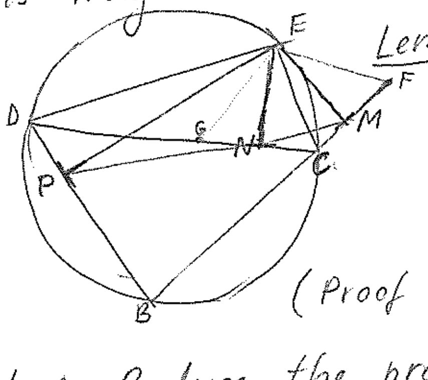
Strategy: Using $AB \parallel CD$ and $AD \parallel BC$, we can see that
AE = angle bisector of $\widehat{BAD} \Leftrightarrow \widehat{CFG} = \widehat{CGF} \Leftrightarrow |CF| = |CG|$.
Among all data in the statement, this fits in well with $|FE| = |EG|$ $\Rightarrow$

So we have to prove $\boxed{EC = \perp \text{ bisector of } FG.}$ $\Leftrightarrow \Delta EFC \cong \Delta EGC$.

However, we want to use mostly angles (parallelism, cyclicity), and in particular angles on the circle.
$\Rightarrow$ proving $\boxed{\widehat{DCE} = \widehat{ECG}}$ seems like a good strategy, because we can now use arcs on the circle.

On the other hand, we haven't used fully $|EF| = |EC| = |EG|$. This tells us that E = circumcentre of ΔFCG, so intersection of $\perp$ bisectors.
Draw $EM \perp CG$ and $EN \perp CF$. (Draw a larger diagram!)

This may remind us of Simpson's line:

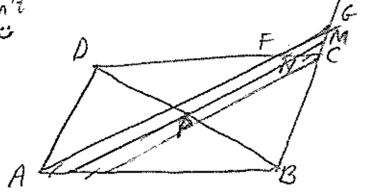


Lemma

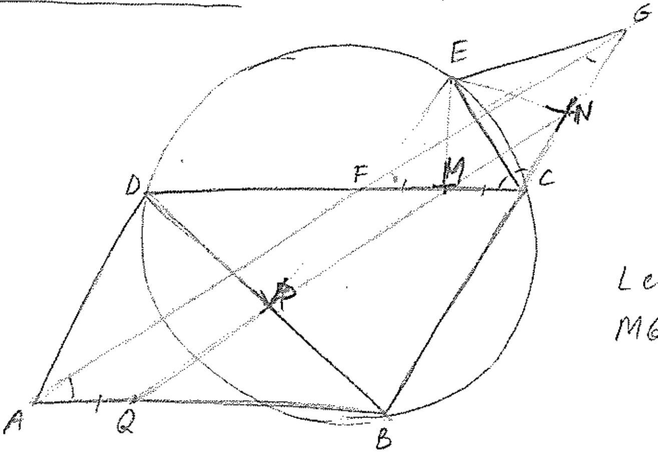
$BCDE$ cyclic, $EP \perp BD$, $EN \perp CD$, $EM \perp BC$ $\Rightarrow$ M, N, P collinear

(Proof of lemma → Ex. set. 3: Use angles and cyclic quadrilat.)

Hint: Reduce the problem $\widehat{DCE} = \widehat{ECG}$ to proving that P is the midpoint of BD, Even if it doesn't look like it ☺. Then use the parallelogram properties:



IMO 2007 Problem 2 Solution:



Let M = midpoint of FC

N = midpoint of GC.

MN = midline in $\triangle CGF \Rightarrow MN \parallel FG$

Let MN intersect BD at P and AB at Q.

$MQAF$ parallelogram $\Rightarrow |FM| = |AQ|$. $\left.\right\} \Rightarrow |AQ| = |CM|$

But $|FM| = |CM|$

$\Rightarrow |DM| = |QB| = |AB| - |AQ|$ $\left.\right\} \Rightarrow$ $DMBQ$ parallelogram

$DM \parallel QB$

$\Rightarrow MQ$ intersects BD at midpoint P.

On the other hand, MNP is Simpson's line for the point E and $\triangle BCD$ (meaning that EM, EN, EP are $\perp$-s on the sides of $\triangle BCD$). $\Rightarrow EP \perp BD$ and P = midpoint of $BD \Rightarrow EP = \perp$ bisector of BD.

$\Rightarrow \widehat{DE} = \widehat{BE}$. But $\widehat{ECD} = \dfrac{\widehat{DE}}{2}$

$\widehat{ECG} = 180^\circ - \widehat{ECB} = 180^\circ - \dfrac{\widehat{FDB}}{2} = \dfrac{\widehat{EB}}{2}$ $\left.\right\} \Rightarrow \widehat{ECD} = \widehat{ECG}$.

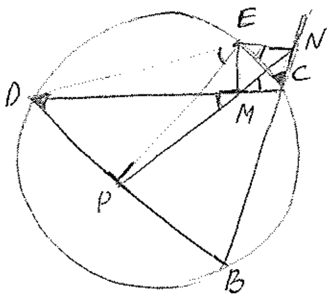
$\Rightarrow \triangle EFC \equiv \triangle EGC$ (isosceles triangles, EC common size and $\widehat{ECD} = \widehat{ECG}$)

$\Rightarrow |CF| = |CG| \Rightarrow \widehat{CGF} = \widehat{CFG} = \widehat{FAB}$ (corresponding angles) $\left.\right\} \Rightarrow$

Also, $\widehat{CGF} = \widehat{FAD}$ (alternate angles)

$\Rightarrow \boxed{\widehat{FAB} = \widehat{FAD}}$ q.e.d

Proof of Simpson's Lemma:



We know:

$EM \perp BC$

$EN \perp CD$

To prove: $EP \perp BD \Leftrightarrow M, N, P$ collinear.

Proof Note "$\Rightarrow$" is sufficient, as the line MN intersects BD at exactly one point.

Proof of "$\Rightarrow$": $BCED$ cyclic $\Rightarrow \widehat{BDE} = \widehat{NCE}$

$\widehat{EPD} = \widehat{ENC} = 90^\circ$ $\left.\right\} \Rightarrow \triangle EPD \sim \triangle ENC$

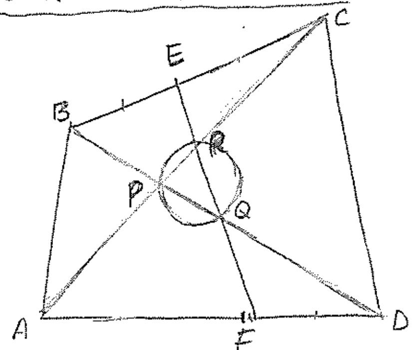
$\Rightarrow \widehat{DEP} = \widehat{CEN}$ $\left.\right\} \Rightarrow \widehat{DMP} = \widehat{NMC}$.

But $\widehat{EMD} = \widehat{EPD} = 90^\circ \Rightarrow EMPD$ cyclic $\Rightarrow \widehat{DEP} = \widehat{DMP}$

Similarly $EMCN$ cyclic $\Rightarrow \widehat{CEN} = \widehat{NMC}$

D, M, C collinear, N, M on opposite sides of $CD \Rightarrow N, M, P$ collinear.

IMO 2005 Problem 5



We know:

$|BC| = |AD|$ and $BC \not\parallel AD$

$|BE| = |DF|$

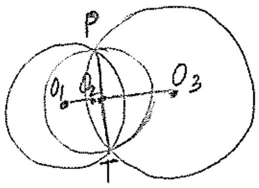
To prove: As E varies on BC (and F on AD with $|BE| = |DF|$)

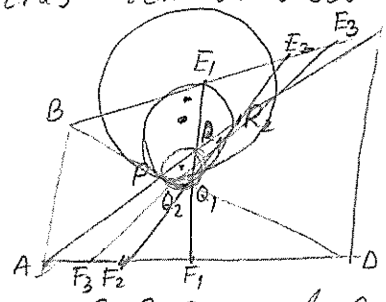
$\mathscr{C}(PRQ)$ passes through 2 fixed points P and T.

(circumcircle of $\triangle PRQ$)

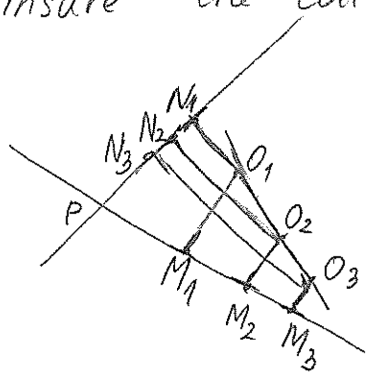
Strategy: Unfortunately, it seems that the problem invites us to draw a complicated diagram by considering $\mathscr{C}(PRQ)$ for various positions of E on BC. Also unfortunately, there seems to be no relevant information about $\mathscr{C}(PRQ)$ coming from the given data. On the contrary, the given data consists exclusively of equal segments. This suggests strategies like congruences, Thales, Menelaus' theorems: segment-based strategies. This means, that from $\mathscr{C}(PRQ)$ we should remember the circumcentre and the radius, and can safely forget about arcs and angles.

Consider E_1, E_2, E_3 on BC, F_1, F_2, F_3 on AD so that $|BE_1| = |DF_1|$, $|BE_2| = |DF_2|$, $|BE_3| = |DF_3|$. Suppose $\mathscr{C}(PR_1Q_1)$, $\mathscr{C}(PR_2Q_2)$, $\mathscr{C}(PR_3Q_3)$ all have 2 points in common. What does this tell us about their circumcentres O_1, O_2, O_3?





Hint ① Focus on the points O_1, O_2, O_3, Q_1, Q_2, Q_3 and R_1, R_2, R_3. Can you write a relation between $Q_1, Q_2, Q_3, R_1, R_2, R_3$ which would insure the collinearity of O_1, O_2, O_3? Strategy:



Let M_1 = midpoint of OQ_1,

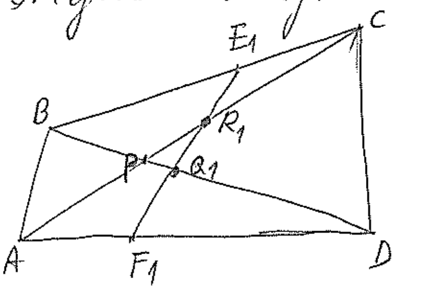
N_1 = midpoint of OR_1,

etc.

Find a relation between M_1, M_2, M_3 and N_1, N_2, N_3 first. Work with ratios of segments.

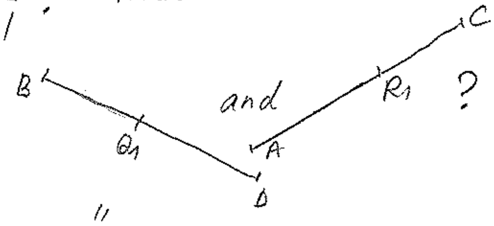
How does M_1M_2 compare to Q_1Q_2?
N_1N_2 to R_1R_2?
Now find a relation between R_1, R_2, R_3 and Q_1, Q_2, Q_3.

Hint ② As it's rather complicated to include all points E_i, F_i, Q_i, R_i in the diagram, focus on the original diagram for a time:



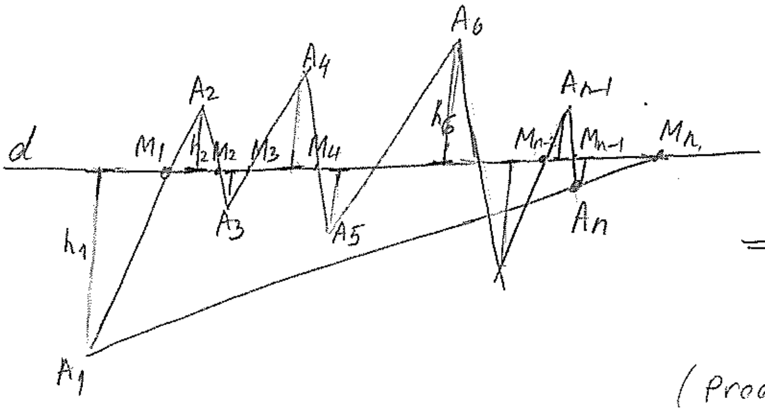
$|CE_1| = |AF_1|$, $|BE_1| = |CE_1|$.

$\Rightarrow \dfrac{|BE_1|}{|CE_1|} = \dfrac{|AF_1|}{|DF_1|}$. What does this tell us about



?

Hint ③ Use "<u>Menelaus' generalized Theorem</u>":



Polygon $A_1A_2A_3\ldots A_n$ crosses line d at $M_1, M_2, M_3, \ldots, M_n$

$$\Longrightarrow \boxed{1 = \frac{|A_1M_1|}{|M_1A_2|} \cdot \frac{|A_2M_2|}{|M_2A_3|} \cdot \ldots \cdot \frac{|A_{n-1}M_{n-1}|}{|M_{n-1}A_n|} \cdot \frac{|A_nM_n|}{|M_nA_1|}}$$

$$\left(\text{Proof}: 1 = \frac{h_1}{h_2} \cdot \frac{h_2}{h_3} \cdot \ldots \cdot \frac{h_{n-1}}{h_n} \cdot \frac{h_n}{h_1}\right).$$

(Apply this theorem to BCAD crossed by line E_1F_1)

Hint ③: From the equation: $\dfrac{|CR_1|}{|R_1A|} = \dfrac{|BQ_1|}{|Q_1D|}$

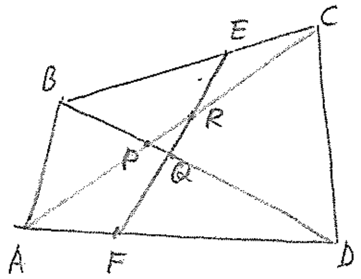
derive other similar equations, but where some of the terms are fixed (contain no R_1, Q_1).

Use <u>algebra tricks</u>: $\boxed{\dfrac{x}{y} = \dfrac{x'}{y'} \Rightarrow \dfrac{x}{x \pm y} = \dfrac{x'}{x' \pm y'}}$

Combine the resulting equations for $|CR_1|$, $|BQ_1|$ with similar equations for $|CR_2|$, $|BQ_2|$, to get info on $|R_1R_2|$, $|Q_1Q_2|$. similarly for $|R_2R_3|$, $|Q_2Q_3|$.

<u>Trick</u>: $\boxed{\dfrac{x}{y} = \dfrac{x'}{y'} \Rightarrow \dfrac{x}{y} = \dfrac{x'}{y'} = \dfrac{x \pm x'}{y \pm y'}}$

IMO 2005 Problem 5   Solution:

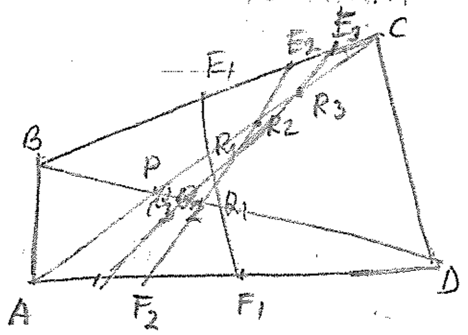


Generalized Menelaus' theorem for $BCAD$ crossed by line EF

$$\Rightarrow \frac{|BE|}{|EC|}\cdot\frac{|CR|}{|RA|}\cdot\frac{|AF|}{|FD|}\cdot\frac{|DQ|}{|QB|}=1$$

But $\frac{|BE|}{|EC|}=\frac{|FD|}{|AF|} \Rightarrow \frac{|CR|}{|RA|}=\frac{|BQ|}{|QD|}$.

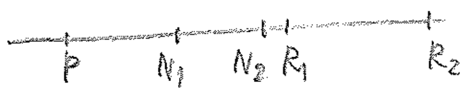
$$\Rightarrow \frac{|CR|}{|CR|+|RA|}=\frac{|BQ|}{|BQ|+|QD|} \Rightarrow \frac{|CR|}{|CA|}=\frac{|BQ|}{|BD|} \Rightarrow \boxed{\frac{|CR|}{|BQ|}=\frac{|CA|}{|BD|}}$$



$$\Rightarrow \frac{|CR_1|}{|BQ_1|}=\frac{|CR_2|}{|BQ_2|}=\frac{|CA|}{|BD|} \Rightarrow \boxed{\frac{|R_1R_2|}{|Q_1Q_2|}=\frac{|CA|}{|BD|}}$$

Similarly $\boxed{\frac{|R_2R_3|}{|Q_2Q_3|}=\frac{|CA|}{|BD|}}$

$$\Rightarrow \boxed{\frac{|R_1R_2|}{|R_2R_3|}=\frac{|Q_1Q_2|}{|Q_2Q_3|}}.$$



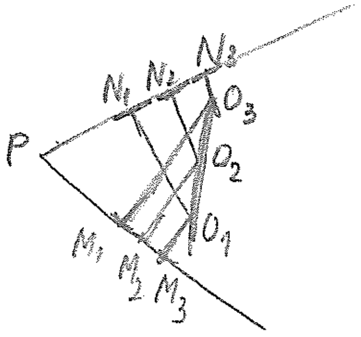
Let N_1 = midpoint of PR_1
N_2 = midpoint of PR_2.

$$\Rightarrow |N_1N_2| = |PN_2|-|PN_1| = \frac{|PR_2|-|PR_1|}{2}=\frac{|R_1R_2|}{2}$$

Let M_i = midpoint of PQ_i $\Rightarrow |M_1M_2|=\frac{|Q_1Q_2|}{2}$.

similarly for $|M_2M_3|$, $|N_2N_3|$

$$\Rightarrow \boxed{\frac{|N_1N_2|}{|N_2N_3|}=\frac{|M_1M_2|}{|M_2M_3|}}$$



$O_iM_i \perp BD$ and $O_iN_i \perp AC$

parallel   parallel.

Looking at the intersection of O_2M_2 and O_2N_2 with O_1O_3

$\Rightarrow O_1, O_2, O_3$ collinear

$\Rightarrow \mathcal{C}(PR_1Q_1)$, $\mathcal{C}(PR_2Q_2)$, $\mathcal{C}(PR_3Q_3)$ all intersect at 2 points.