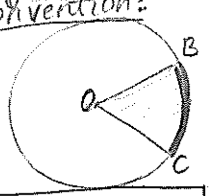
Review: Basic Tools 1

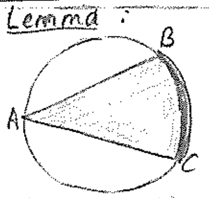
Angles and arcs:

Convention:



$\boxed{\underset{\text{radians}}{\widehat{BOC}} = \underset{\text{radiuses}}{\widehat{BC}}}$

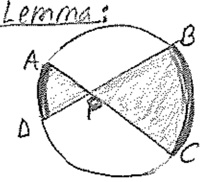
Lemma:



Angle on the circle:

$\widehat{BAC} = \dfrac{\widehat{BOC}}{2} = \dfrac{\widehat{BC}}{2}$

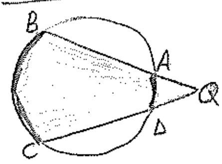
Lemma:



Interior angle:

$\widehat{APD} = \widehat{BPC} = \dfrac{\widehat{AD} + \widehat{BC}}{2}$

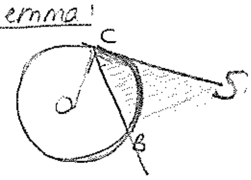
Lemma:



Exterior angle:

$\widehat{BQC} = \dfrac{\widehat{BC} - \widehat{AD}}{2}$

Lemma:

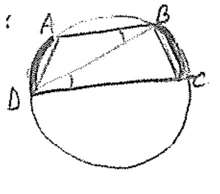


Angle between tangent and chord:

$\widehat{SCB} = \dfrac{\widehat{BC}}{2}$

Arcs and chords:

Lemma:



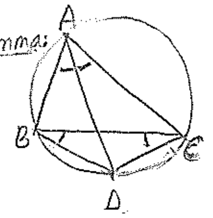
The following are equivalent:

① $\widehat{AD} = \widehat{BC}$

② $|AD| = |BC|$

③ $AB \parallel CD$

Lemma:



The following are equivalent:

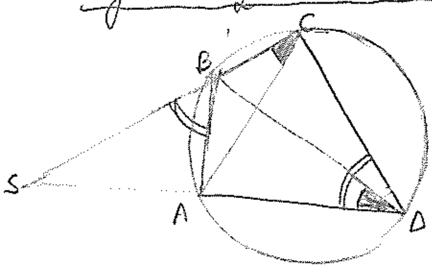
① AD = angle bisector of $\widehat{BAC}$

② $\widehat{BD} = \widehat{DC}$

③ $|BD| = |DC|$.

④ $\widehat{BAD} = \widehat{DAC} = \widehat{DBC} = \widehat{DCB} = \frac{1}{2}\widehat{A}$.

Cyclic quadrilaterals:



Theorem:

The following are equivalent:

① $ABCD$ cyclic

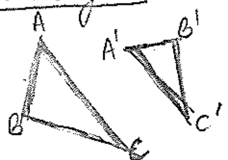
② $\widehat{ABC} + \widehat{ADC} = 180°$

③ $\widehat{SBA} = \widehat{ADC}$

④ $\widehat{BCA} = \widehat{BDA}$

Similar triangles

Definition:



$\boxed{\Delta ABC \sim \Delta A'B'C'} \Longleftrightarrow \begin{cases} \widehat{A} = \widehat{A'} \\ \widehat{B} = \widehat{B'} \\ \widehat{C} = \widehat{C'} \end{cases}$

(note: 2 pairs are sufficient)

Theorem

$\Delta ABC \sim \Delta A'B'C'$

$\Longleftrightarrow \boxed{\dfrac{|AB|}{|A'B'|} = \dfrac{|AC|}{|A'C'|} = \dfrac{|BC|}{|B'C'|}}$

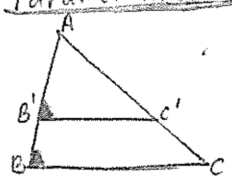
(sides are proportional)

Theorem (S.A.S. for similar triangles)

$\Delta ABC \sim \Delta A'B'C'$

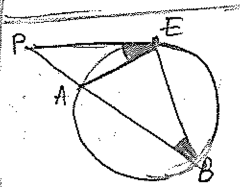
$\Longleftrightarrow \begin{cases} \dfrac{|AB|}{|A'B'|} = \dfrac{|AC|}{|A'C'|} \text{ AND} \\ \widehat{A} = \widehat{A'} \end{cases}$

Parallel lines



$\Delta ABC \sim \Delta AB'C'$

$\Leftrightarrow \dfrac{|AB|}{|AB'|} = \dfrac{|AC|}{|AC'|} = \dfrac{|BC|}{|B'C'|}$

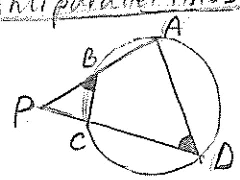


PE tangent

$\Leftrightarrow \Delta PAE \sim \Delta PEB$

$\Leftrightarrow \dfrac{|PA|}{|PE|} = \dfrac{|PE|}{|PB|}$

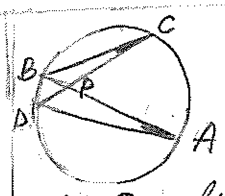
Antiparallel lines



$ABCD$ cyclic

$\Leftrightarrow \Delta PBC \sim \Delta PDA$

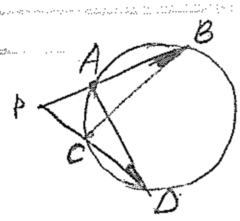
$\Leftrightarrow \dfrac{|PB|}{|PD|} = \dfrac{|PC|}{|PA|}$



$ADBC$ cyclic

$\Leftrightarrow \Delta PBC \sim \Delta PDA$

$\Leftrightarrow \dfrac{|PB|}{|PD|} = \dfrac{|PC|}{|PA|}$

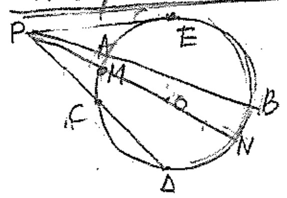


$ABDC$ cyclic

$\Leftrightarrow \Delta PBC \sim \Delta PDA$

$\Leftrightarrow \dfrac{|PB|}{|PD|} = \dfrac{|PC|}{|PA|}$

The power of a point P with respect to circle $\mathcal{C}$:



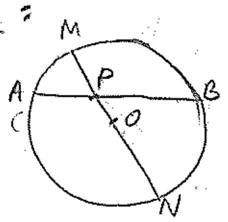
$|PA| \cdot |PB| = |PE|^2 = |PC| \cdot |PD|$

= power of P

= the same on any line through P

$= |PO|^2 - \text{radius}^2$

$= (|PM| \cdot |PN| = (|PO| - \text{radius})(|PO| + \text{radius})$



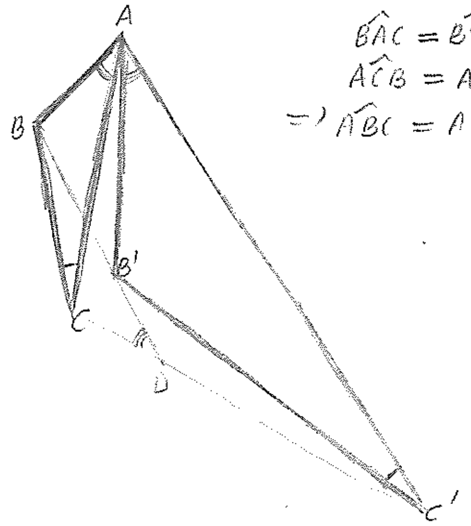
Power of P

$= |PA| \cdot |PB|$

$= \text{radius}^2 - |OP|^2$

$(= |PM| \cdot |PN|)$

Another common type of similar triangles: The "rotated" similar triangles



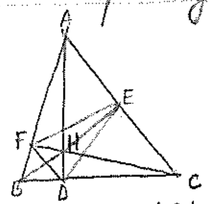
$\widehat{BAC} = \widehat{B'AC'}$
$\widehat{ACB} = \widehat{AC'B'}$
$\Rightarrow \widehat{ABC} = \widehat{AB'C'}$ $\Big\}$ $\boxed{\Delta ABC \sim \Delta AB'C'}$

$\Rightarrow \dfrac{|AB|}{|AB'|} = \dfrac{|AC|}{|AC'|}$
$\widehat{BAB'} = \widehat{BAC} + \widehat{CAB'} = \widehat{CAC'}$ $\Big\} \Rightarrow \boxed{\Delta ABB' \sim \Delta ACC'}$

$\Rightarrow \widehat{ABB'} = \widehat{ACC'}$
and $\widehat{AB'B} = \widehat{AC'C}$

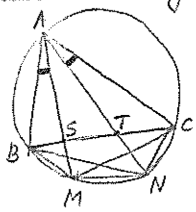
Also, $AB'DC'$ cyclic, $ABCD$ cyclic
$\widehat{BCD} = \widehat{BAC} = \widehat{B'AC'}$

Similar triangles are THE most useful tool in solving IMO geometry problems, because they help you combine angle-based strategies with segment-based strategies. Practice spotting pairs of similar triangles, for example:



$AD \perp BC$
$BE \perp AC$
$CF \perp AB$

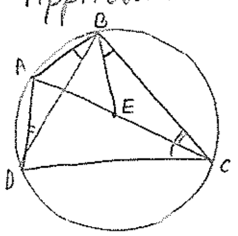
There are at least 24 pairs here! ←



$\widehat{BAM} = \widehat{CAN}$

There are at least 4 pairs here.
(Using labelled points).

Application:



Ptolemy's Theorem:
$\boxed{|AB| \cdot |CD| + |AD| \cdot |BC| = |AC| \cdot |BD|}$ *

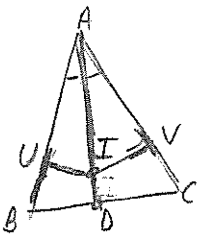
Proof: Construct BE such that $\widehat{EBC} = \widehat{ABD}$

Then $\Delta ABD \sim \Delta EBC \Rightarrow \dfrac{|AD|}{|CE|} = \dfrac{|BD|}{|BC|} \Rightarrow |AD| \cdot |BC| = |BD| \cdot |CE|$

"rotated" Δ-s ⇒ Also then $\Delta ABE \sim \Delta DBC \Rightarrow \dfrac{|AE|}{|CD|} = \dfrac{|AB|}{|BD|} \Rightarrow |AB| \cdot |CD| = |BD| \cdot |AE|$

Add up to *.

Other useful methods connecting angles with segments:

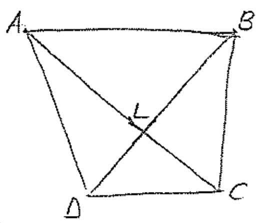


AI = angle bisector of $\widehat{BAC}$
$\Leftrightarrow |IU| = |IV|$ where $IU \perp AB$, $IV \perp AC$
(Don't be afraid to add IU, IV to diagrams of angle bisectors).

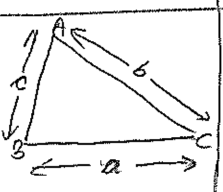
Also, $\boxed{\dfrac{|BD|}{|DC|} = \dfrac{|AB|}{|AC|} = \dfrac{Area(ABD)}{Area(ACD)}}$

Pythagora!

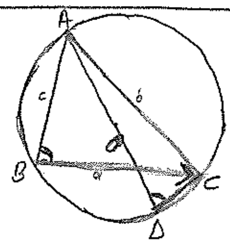
$AC \perp BD$
$\Leftrightarrow \boxed{|AD|^2 - |AB|^2 = |CD|^2 - |CB|^2}$
$(= |LD|^2 - |LB|^2)$



Cos rule: $\boxed{a^2 = b^2 + c^2 - 2bc \cdot \cos A}$

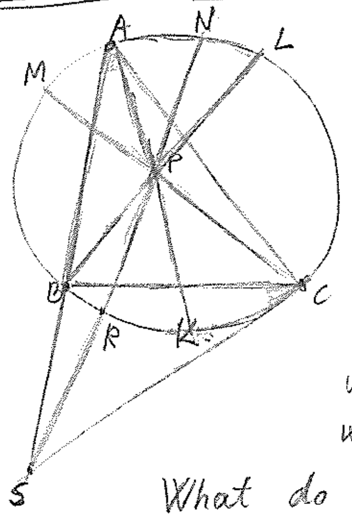


sin rule: In Δ ACD,
$\widehat{ACD} = 90°$, $\widehat{ADC} = \widehat{B}$
$\Rightarrow \dfrac{b}{2R} = \sin \widehat{B}$
$\Rightarrow 2R = \dfrac{b}{\sin B} = \dfrac{a}{\sin A} = \dfrac{c}{\sin C}$



IMO 2010 Problem 4

Diagram Hint: Draw circle first, then SC tangent, then $|SP|=|SC|$, then the other elements.



We know:

- SC tangent to circle $\mathscr{C}(ABC)$
- $|SP| = |SC|$

To prove:

$|MK| = |ML|$.

Strategy: Since M, K, L are on the circle $\mathscr{C}(ABC)$, we could try to prove $\overset{\frown}{MK} = \overset{\frown}{ML}$. That is, work with angles and corresponding arcs on $\mathscr{C}(ABC)$.

What do the two facts we know tell us about angles ... and arcs?

Hint ① $|SP| = |SC| \Rightarrow \widehat{SPC} = \widehat{SCP}$.

In order to write $\widehat{SPC}$ in terms of arcs, clearly we need to intersect the line SP with the circle. Call the two intersection points R and N, like in the diagram. Now write $\widehat{SPC}$ and $\widehat{SCP}$ in terms of arcs and compare the results.

Can you simplify? Compare to our goal $\overset{\frown}{MK} = \overset{\frown}{ML}$.

What remains to prove?

Hint ② Revert back to angles so that you can prove $\overset{\frown}{NL} = \overset{\frown}{RK}$.

Note that no angle in the picture corresponds to $\overset{\frown}{NL}$, but

$\widehat{NPL} = \widehat{BPR} = \dfrac{\overset{\frown}{NL}}{2} + \dfrac{\overset{\frown}{BR}}{2}$.

Compare to $\dfrac{\overset{\frown}{RK}}{2} + \dfrac{\overset{\frown}{BR}}{2} = \widehat{BAP}$

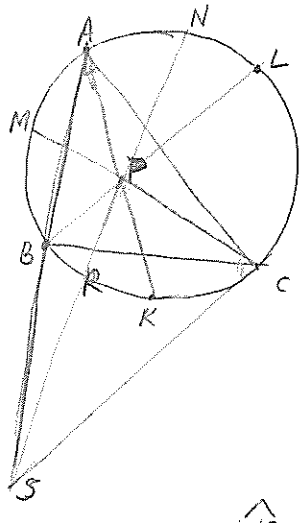
Can you prove that these angles are equal?

Hint ③ Find a pair of similar triangles containing the angles $\widehat{BAP}$ and $\widehat{BPS}$. We never used the points A, B until now, and the fact that AB goes through S, just like SC.

Use that now!

Hint ④ Use the power of the point S and the fact that $|SP| = |SC|$ to prove that the following triangles are similar:

$\Delta SBP \sim \Delta SPA$ by the case S.A.S.



Solution :

$|SP| = |SC| \Rightarrow \widehat{SPC} = \widehat{SCP}$ (ΔSPC isosceles)

$\left.\begin{array}{l}SC \text{ tangent} \\ CM \text{ chord}\end{array}\right\} \Rightarrow \widehat{SCP} = \dfrac{\overset{\frown}{CM}}{2}$

$\widehat{SPC} = \dfrac{\overset{\frown}{RC}}{2} + \dfrac{\overset{\frown}{MN}}{2}$ (internal angle)

$\Rightarrow \overset{\frown}{CM} = \overset{\frown}{RC} + \overset{\frown}{MN}$

But $\overset{\frown}{CM} = \overset{\frown}{RC} + \overset{\frown}{RM}$ $\Big\} \Rightarrow \overset{\frown}{RM} = \overset{\frown}{MN}$ ①

Remains to prove : $\overset{\frown}{RK} = \overset{\frown}{NL}$ ②

Now $\left.\begin{array}{l}\widehat{NPL} = \dfrac{\overset{\frown}{NL}}{2} + \dfrac{\overset{\frown}{BR}}{2} = \widehat{BPR} \\ \dfrac{\overset{\frown}{RK}}{2} + \dfrac{\overset{\frown}{BR}}{2} = \widehat{BAK}\end{array}\right\}$ Remains to prove: $\widehat{BAK} = \widehat{BPR}$.

Note that $\left.\begin{array}{l}\widehat{SAC} = \widehat{SCB} = \dfrac{\overset{\frown}{BS}}{2} \\ \widehat{ASC} = \widehat{CSB} \text{ (same } \measuredangle)\end{array}\right\} \Rightarrow \Delta SAC \sim \Delta SCB$

$\Rightarrow \dfrac{|SA|}{|SC|} = \dfrac{|SC|}{|SB|} \Rightarrow |SC|^2 = |SA| \cdot |SB|$ (the "power of point"s) $\Big\} \Rightarrow$

But $|SC| = |SP|$ (given)

$\Rightarrow |SP|^2 = |SA| \cdot |SB|$.

$\Rightarrow \left.\begin{array}{l}\dfrac{|SA|}{|SP|} = \dfrac{|SP|}{|SB|} \\ \widehat{ASP} = \widehat{PSB} \text{ (same angle)}\end{array}\right\} \overset{S.A.S.}{\Rightarrow} \Delta ASP \sim \Delta PSB$

$\Rightarrow \widehat{BAP} = \widehat{BPS}$

$\Rightarrow \dfrac{\overset{\frown}{RK} + \cancel{\overset{\frown}{BR}}}{\cancel{2}} = \dfrac{\overset{\frown}{NL} + \cancel{\overset{\frown}{BR}}}{\cancel{2}}$

$\Rightarrow \overset{\frown}{RK} = \overset{\frown}{NL}$ ②

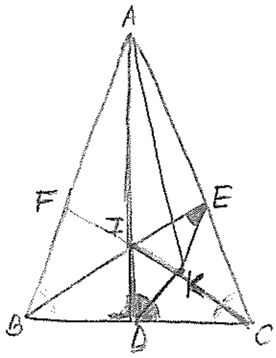
Also $\overset{\frown}{RM} = \overset{\frown}{MN}$ ①

Add $\Rightarrow \overset{\frown}{MK} = \overset{\frown}{ML}$.

$\Rightarrow |MK| = |ML|$ ∎

(because $\Delta MOK \equiv \Delta MOL$
where O = circumcentre of $\mathcal{C}(ABC)$)

IMO 2009   Problem 4:



We know :
- $|AB| = |AC|$
- AD = angle bisector of $\widehat{BAC}$.
- BE = angle bisector of $\widehat{ABC}$,
- DK = angle bisector of $\widehat{ADC}$.
- CK = angle bisector of $\widehat{ACD}$.
- $\widehat{IEK} = 45°$.

To calculate : $\widehat{CAB}$.

Note: The problem is equivalent to finding $\widehat{ACD}$, since $\widehat{BAC} = 180° - 2\widehat{ACD}$. $\widehat{ACD}$ seems more convenient as it's "closer to the action".

Also Note: ΔABC isosceles, AD angle bisector $\Rightarrow$ AD symmetry axis for ΔABC. The angle bisector of $\widehat{ACB}$ contains both the incentres of ΔACD and ΔABC, so CK passes through I. Extend CI to meet AB at F, for symmetry!

Possible Strategies: I Angle Calculations

II Angle calculations $\Rightarrow$ Similar triangles $\Rightarrow$ segment calculations.

Strategy I As $AD \perp ADC$ and DK is the angle bisector of $\widehat{ADC}$, then

$\widehat{IDK} = \widehat{KDC} = \widehat{IEK} = 45°$.

At this point, it is so tempting to try and prove $\begin{cases} \Delta IDC \equiv \Delta IEC \\ \Delta IDK \equiv \Delta IEK \end{cases}$

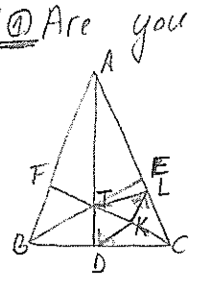
But WE CANNOT, because $\widehat{IDC} = 90°$ but we don't know $\widehat{IEC}$!

If $\widehat{IEC}$ was $90°$, then $BE \perp AC$ so ΔABC - equilateral!

So → Step I Check that ΔABC equilateral $\Rightarrow$ $\widehat{IEK} = 45°$.

→ Step II Deal with the case ΔABC is NOT equilateral.

Hint ① Are you very annoyed that $\widehat{IEC} \neq 90°$? Good! Act on it!

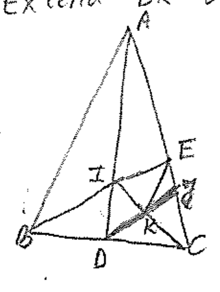


Construct your very own $IL \perp AC$ and then note that CI is symmetry line for $\Delta IDC \equiv \Delta ILC$. Can you find other $45°$ angles in the diagram? (Draw a large diagram unlike mine). Can you find a cyclic quadrilater? $\Rightarrow$ other equal angles? What does this tell you about $\widehat{ABC}$?

Hint ② Try proving $DK \parallel BF$.

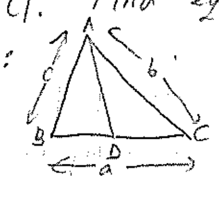
Hint ③ Use $\widehat{KDC} = 45°$!

Strategy II Is not for the faint of heart, as it uses lots of calculations! Calculate all angles in terms of $\alpha = \frac{\widehat{ACB}}{2}$. Extend DK to meet AC at J.



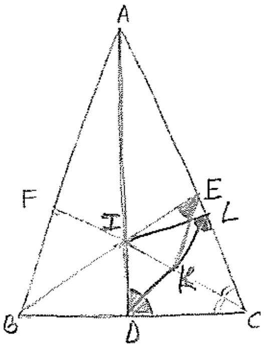
Hint ① $\Delta EIK \sim \Delta DCJ$. Calculate $|EI|, |IK|, |DC|, |CJ|$ in terms of $a = |BC|$ and $b = |AC|$. Find equation in a and b.

Hint ② Use Lemma:



AD angle bisector of $\widehat{BAC}$ $\Rightarrow$ $|BD| = \frac{ac}{b+c}$, $|CD| = \frac{ab}{b+c}$.

IMO 2009 Problem 4



Solution I First note: $\left.\begin{array}{l}\triangle ABC \text{ isosceles,} \\ AD \text{ angle bisector}\end{array}\right\} \Rightarrow AD \perp BC$

$\Rightarrow \widehat{ADC} = 90° \Rightarrow \widehat{IDK} = \widehat{KDC} = 45° = \widehat{IEK}$!

Case I) if $\widehat{KEC} = 45°$ too, so $IE \perp EC$. Then $\triangle ABC$ would be equilateral, ($\Leftrightarrow \boxed{\widehat{BAC} = 60°}$) since AD and BE would be both angle bisectors and altitudes (heights) in $\triangle ABC$.

Indeed, $\triangle ABC$ equilateral $\Rightarrow \triangle ABE \equiv \triangle CBE \Rightarrow BE \perp AC$

and $\triangle CDK \equiv \triangle CEK$ because S.A.S. $\left\{\begin{array}{l}|CK| = |CK| \\ \widehat{DCK} = \widehat{KCE} \text{ (CK angle bisector)} \\ |CD| = |CE| = \frac{1}{2} \text{ side}\end{array}\right.$

$\Rightarrow \widehat{CEK} = \widehat{CDK} = 45°$

$\Rightarrow \widehat{IEK} = 90° - 45° = 45°$

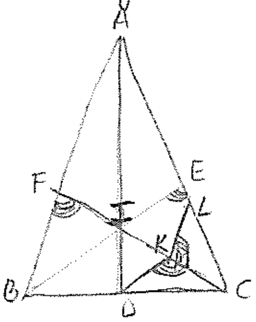
Case II) If $\widehat{KEC} \neq 45°$ so $\widehat{IEC} \neq 90°$, so $\triangle IEC \not\equiv \triangle IDC$. Then we draw $IL \perp AC$ to make our own pair of congruent triangles:

$\left.\begin{array}{l}\widehat{ILC} = \widehat{IDC} = 90° \\ |IC| = |IC| \\ \widehat{DCI} = \widehat{ECI} \\ \text{(CI = angle bisector)}\end{array}\right\}$ S.A.A. $\boxed{\triangle ILC \equiv \triangle IDC}$

$\left.\begin{array}{l}\Rightarrow |LC| = |DC| \\ \widehat{LCK} = \widehat{DCK} \\ |CK| = |CK|\end{array}\right\}$ S.A.S. $\Rightarrow \boxed{\triangle LCK \equiv \triangle DCK}$

$\left.\begin{array}{l}\Rightarrow \widehat{CDK} = 45° = \widehat{CLK} \\ \text{Also, } IL \perp AC\end{array}\right\} \Rightarrow \boxed{\widehat{ILK} = 45°}$

Using $\widehat{IEK} = 45° = \widehat{ILK} \Rightarrow IELK$ cyclic quadrilateral



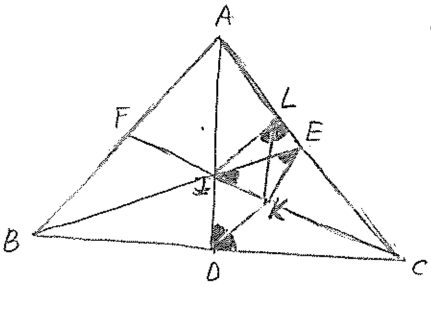
$\left.\begin{array}{l}\Rightarrow \widehat{IEC} = \widehat{LKC}. \\ \text{But } \widehat{IEC} = \widehat{IFB} \text{ by symmetry} \\ (\triangle ABC \text{ isosceles with } AD = \text{symmetry axis}) \\ \text{and } \widehat{LKC} = \widehat{DKC} \\ \text{(because } \triangle LCK \equiv \triangle DCK)\end{array}\right\} \Rightarrow \widehat{IFB} = \widehat{DKC}$

$\Rightarrow \boxed{DK \parallel BF}$

$\Rightarrow \widehat{KDC} = \widehat{ABC}$

But $\widehat{KDC} = 45° \Rightarrow \widehat{ABC} = \widehat{ACB} = 45° \Rightarrow \boxed{\widehat{BAC} = 90°}$

In this case, the diagram would actually look like this:



Calculating all possible angles in the figure:

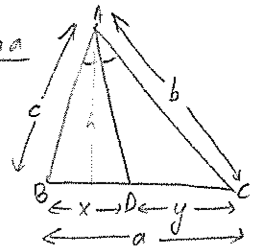
$\left.\begin{array}{l}\widehat{IDK} = \widehat{KDC} = 45° = \widehat{ILK} = \widehat{KLC} \text{ (as } \triangle LCK \equiv \triangle DCK) \\ \widehat{EIC} = \widehat{IBC} + \widehat{ICB} = \frac{1}{2}(\widehat{ABC} + \widehat{ACB}) = 45°\end{array}\right\} \Rightarrow$

$\Rightarrow IKEL$ cyclic

$\Rightarrow \widehat{IEK} = \widehat{ILK} = 45°.$

JMO 2009 Problem 4 Solution II:

Lemma



AD angle bisector of $\widehat{BAC}$

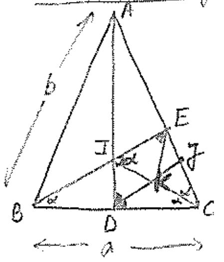
$$\Rightarrow \boxed{\frac{|BD|}{|CD|} = \frac{c}{b}} \Rightarrow \boxed{|BD| = \frac{c \cdot a}{b+c}}, \quad \boxed{|CD| = \frac{b \cdot a}{b+c}}$$

Proof of Lemma: $\begin{cases} \dfrac{x}{y} = \dfrac{Area(\Delta ABD)}{Area(\Delta ACD)} = \dfrac{\cancel{\frac{1}{2}}|AB|\cdot\cancel{|AD|}\cdot \cancel{\sin\frac{A}{2}}}{\cancel{\frac{1}{2}}|AC|\cdot\cancel{|AD|}\cdot \cancel{\sin\frac{A}{2}}} = \dfrac{c}{b} \\ x+y = a. \end{cases}$

Solve the system by substitution $\Rightarrow x = \dfrac{ca}{b+c}, \quad y = \dfrac{ba}{b+c}.$

(Note: Hence $|AD|$ can be calculated by $\cos B$ formula in ΔABD)

Solution of problem 4.



We wish to calculate $\alpha = \dfrac{\widehat{ACB}}{2}$. Note that in ΔADC, $\boxed{\cos 2\alpha = \dfrac{a}{2b}}$.

Let J be the intersection of DK with AC.

similar Δ's:

$$\left.\begin{array}{l} \widehat{EIK} = \widehat{ICB} + \widehat{IBC} = 2\alpha = \widehat{DCJ} \\ \widehat{IEK} = 45^\circ = \widehat{CDJ} \end{array}\right\} \Rightarrow \Delta EIK \sim \Delta DCJ \Rightarrow \boxed{\frac{|EI|}{|CD|} = \frac{|IK|}{|CJ|}} \quad (1)$$

We calculate $|EI|, |CD|, |IK|, |CJ|$ in terms of a, b. Note $\boxed{|CD| = \dfrac{a}{2}}$

- CI angle bisector in $\Delta CBE \Rightarrow \dfrac{|EI|}{|BI|} = \dfrac{|CE|}{|BC|} \Rightarrow |EI| = \dfrac{|BI|\cdot|CE|}{a}$.

- BE angle bisector in $\Delta ABC \Rightarrow \boxed{|CE| = \dfrac{|BC|\cdot|AC|}{|BC|+|AB|} = \dfrac{ab}{a+b}}$.
- In ΔBDI, $|BI|^2 = |BD|^2 + |DI|^2$
- In ΔACD with CI angle bisector, $\left.\begin{array}{l}|DI| = \dfrac{|DC|\cdot|AD|}{|DC|+|AC|} \\ |AD| = \sqrt{|AC|^2 - |CD|^2} = \sqrt{b^2 - \frac{a^2}{4}}\end{array}\right\} \Rightarrow |DI| = \dfrac{\frac{a}{2}\sqrt{b^2-\frac{a^2}{4}}}{\frac{a}{2}+b}$

$$\Rightarrow |BI|^2 = \frac{a^2}{4} + \frac{\frac{a^2}{4}\left(b^2 - \frac{a^2}{4}\right)}{\left(\frac{a}{2}+b\right)^2} \Rightarrow \boxed{|BI| = \frac{a}{2}\sqrt{\frac{2b}{\frac{a}{2}+b}}} \text{ (simplifying)}$$

$$\Rightarrow \boxed{|EI| = \frac{|BI|\cdot|CE|}{a} = \frac{ab}{2(a+b)}\sqrt{\frac{b}{\frac{a}{2}+b}}}$$

- DK angle bisector in $\Delta IDC \Rightarrow |IK| = \dfrac{|DI|\cdot|CI|}{|DI|+|DC|} = \dfrac{\dfrac{\frac{a}{2}\sqrt{b^2-\frac{a^2}{4}}}{\frac{a}{2}+b}\cdot\frac{a}{2}\sqrt{\dfrac{2b}{\frac{a}{2}+b}}}{\dfrac{\frac{a}{2}\sqrt{b^2-\frac{a^2}{4}}}{\frac{a}{2}+b}+\frac{a}{2}}$

$|CI| = |BI|$

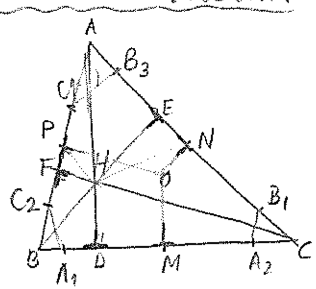
$$\boxed{|IK| = \frac{a}{2}\cdot\frac{\sqrt{2b\left(b-\frac{a}{2}\right)}}{\frac{a}{2}+b+\sqrt{b^2-\frac{a^2}{4}}}}$$

- DJ angle bisector in $\Delta ADC \Rightarrow \boxed{|CJ| = \dfrac{|CD|\cdot|AD|}{|CD|+|AD|} = \dfrac{\frac{a}{2}\cdot b}{\frac{a}{2}+\sqrt{b^2-\frac{a^2}{4}}}}$

Finally, $(1) \Leftrightarrow |EI|\cdot|CJ| = |CD|\cdot|IK| \Leftrightarrow \dfrac{ab}{2(a+b)}\dfrac{\sqrt{b}}{\sqrt{\frac{a}{2}+b}}\cdot\dfrac{\frac{a}{2}\cdot b}{\frac{a}{2}+\sqrt{b^2-\frac{a^2}{4}}} = \dfrac{a}{2}\cdot\dfrac{a}{2}\cdot\dfrac{\sqrt{2b\left(b-\frac{a}{2}\right)}}{\frac{a}{2}+b+\sqrt{b^2-\frac{a^2}{4}}}$

Simplify: $(a^2+ab-2b^2)\left(a-2\sqrt{b^2-\frac{a^2}{4}}\right) = 0 \Leftrightarrow (a+2b)(a-b)\left(a-2\sqrt{b^2-\frac{a^2}{4}}\right) = 0 \Rightarrow \underset{\hat{A}=60^\circ}{\boxed{a=b}}$ or $\underset{\hat{A}=90^\circ}{\boxed{\frac{a}{2} = \sqrt{b^2-\frac{a^2}{4}}}}$

IMO 2008 Problem 1



We know:
$$\begin{cases} H \text{ orthocentre of } \triangle ABC \\ M \text{ midpoint of } BC \\ N \text{ midpoint of } AC \\ P \text{ midpoint of } AB. \\ |HM| = |MA_1| = |MA_2| \\ |HN| = |NB_1| = |NB_2| \\ |HP| = |PC_1| = |PC_2|. \end{cases}$$

To prove: $A_1, A_2, B_1, B_2, C_1, C_2$ all lie on a circle

Strategy: There are 2 possible strategies:

I group $A_1, A_2, B_1, B_2, C_1, C_2$ into quadrilaterals, prove these are cyclic

II Find the candidate for the circumcentre of $A_1A_2B_1B_2C_1C_2$ and prove it's equally distanced from the sides.

The data in the statement of the problem clearly favour Strategy II, because:

ⓐ we've lots of equal segments

ⓑ we've lots of right angles

ⓒ we know virtually nothing about the angles of $A_1A_2B_1B_2$, $A_1A_2B_1C_2$, etc.

ⓐ and ⓑ also strongly suggest using Pythagora's theorem.

Hint ①: The $\perp$ bisectors of A_1A_2, B_1B_2, C_1C_2 are easy to find.

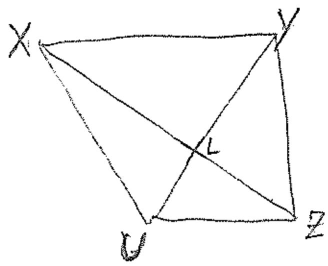
Where do they intersect?

Hint ②: calculate $|OA_1|^2 - |OB_1|^2$ using Pythagora's theorem.

If worst comes to worst, you can calculate all quantities involved in terms of the sizes of $\triangle ABC$ (using the computational techniques in Ex Set 3 on the webpage).

But you can finish faster, for example, use

Lemma

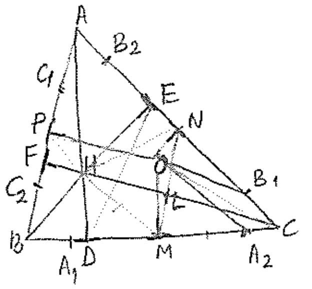


$XZ \perp UY$

$\iff |XU|^2 - |XY|^2 = |ZU|^2 - |ZY|^2$

(because $= |LU|^2 - |LY|^2$).

IMO 2008 Problem 1 Solution:



Since M = midpoint of BC = midpoint of A_1A_2

$\Rightarrow ON = \perp$ bisector of BC = $\perp$ bisector of A_1A_2

Similarly, $ON = \perp$ bisector of $AC = \perp$ bisector of B_1B_2

$OP = \perp$ bisector of $AB = \perp$ bisector of C_1C_2

$\Rightarrow \perp$ bisectors of B_1B_2, C_1C_2, A_1A_2 intersect at O = circumcentre of ΔABC $\Rightarrow$ $|OB_1| = |OB_2|$, $|OC_1| = |OC_2|$, $|OA_1| = |OA_2|$.

It Remains to prove: $|OA_2| = |OB_1|$. Similarly, $|OB_2| = |OC_1|$ and $|OC_2| = |OA_1|$.

Pythagora's theorem: in ΔOMA_2 $\Rightarrow |OA_2|^2 = |OM|^2 + |MA_2|^2 = |OM|^2 + |HM|^2$

in ΔONB_1 $\Rightarrow |OB_1|^2 = |ON|^2 + |NB_1|^2 = |ON|^2 + |HN|^2$

Subtract: $\boxed{|OA_2|^2 - |OB_1|^2 = |OM|^2 - |ON|^2 + |HM|^2 - |HN|^2}$

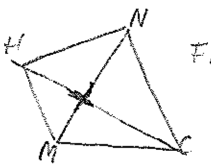
To calculate $|OM|^2 - |ON|^2$: in ΔOMC: $|OM|^2 + |MC|^2 = |OC|^2$

in ΔONC: $|ON|^2 + |NC|^2 = |OC|^2$

Subtract: $\boxed{|OM|^2 - |ON|^2 + |MC|^2 - |NC|^2 = 0}$ $\Rightarrow$

$\Rightarrow$ It Remains to prove: $\boxed{|MC|^2 - |NC|^2 = |HM|^2 - |HN|^2}$.

But $HC \perp MN$ because MN = midline in $\Delta CBA \Rightarrow MN \parallel AB$ and H orthocentre in $\Delta ABC \Rightarrow HC \perp AB$.



From Pythagora's theorem in ΔCLM, ΔCLN, ΔHLM, ΔHLN

$\Rightarrow \boxed{|MC|^2 - |NC|^2 = |HM|^2 - |HN|^2}$

Solution II Starting as in Solution I, remains to prove:

$0 = |OM|^2 - |ON|^2 + |HM|^2 - |HN|^2 = |OM|^2 - |ON|^2 + |HD|^2 + |DM|^2 - |HE|^2 - |EN|^2 \Rightarrow$

But $|HD|^2 = |HC|^2 - |CD|^2$, $|HE|^2 = |HC|^2 - |EC|^2$

It remains to prove $0 = \underbrace{|OM|^2 - |ON|^2}_{-|CM|^2 + |CN|^2} + \cancel{|HC|^2} \underbrace{- |CD|^2 + |DM|^2}_{\substack{(|DM| - |CD|)(|DM| + |CD|) \\ -|CM| \cdot (2|DM| + |CM|)}} - \cancel{|HC|^2} + \underbrace{|EC|^2 - |EN|^2}_{\substack{(|EC| - |EN|)(|EC| + |EN|) \\ |CN|(2|EN| + |CN|)}}$

$\Leftrightarrow 0 = -2|CM| \cdot (|DM| + |CM|) + 2|CN|(|EN| + |CN|)$

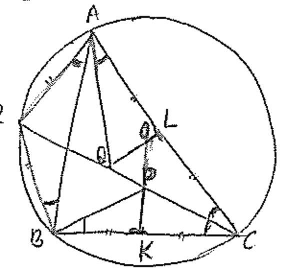
$\Leftrightarrow |CM| \cdot |CD| = |CN| \cdot |CE|$

$\Leftrightarrow ENMD$ cyclic (power of point C).

Indeed, $MN \parallel AB \Rightarrow \widehat{MNC} = \hat{A}$

$\widehat{ADB} = \widehat{AEB} = 90° \Rightarrow AEDB$ cyclic $\Rightarrow \widehat{EDM} = \hat{A}$ $\Big\}$ $\Rightarrow \widehat{MNC} = \widehat{EDM}$ $\Rightarrow ENMC$ cyclic.

IMO 2007 Problem 1



We know:

- RC = angle bisector of $\widehat{ACB}$.
- KP = $\perp$ bisector of BC
- QL = $\perp$ bisector of AC.

To prove:

Area ($\triangle RPK$) = Area ($\triangle RQL$).

Strategy: Just by marking all equal angles due to the angle bisector, cyclic quadrilateral, and $\perp$ bisectors we find plenty of similar triangles. Then try to find some more ☺ Also, compare Area ($\triangle RPK$) with Area ($\triangle RQL$). Use the common line as basis, Do some simplifications untill the problem looks like the consequence of some similar triangles, or possibly, the power of some point(s).

Hints ① $\widehat{RQL} = \widehat{RPK}$. Use sines for the areas.

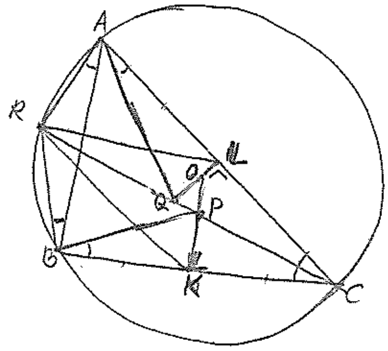
② There are some "rotated $\triangle$-s" in the picture.

②' or, you could reduce the problem to some powers of points: P and Q.

PK intersects QL at circumcentre O which is equally distanced from P and Q.

2007

IMO Problem 1



We know:

- RC = angle bisector of $\widehat{ACB}$
- KP = $\perp$ bisector of BC
- LQ = $\perp$ bisector of AC.
- K = midpoint of BC
- L = midpoint of AC

To prove:

$\text{Area}(\Delta RPK) = \text{Area}(\Delta RQL)$

Solution I: ΔRPK and ΔRQL share a common line RC. We calculate areas based on that line:

$\text{Area}(\Delta RPK) = \frac{1}{2}|RP|\cdot|PK|\cdot \sin \widehat{RPK}$

$\text{Area}(\Delta RQL) = \frac{1}{2}|RQ|\cdot|QL|\cdot \sin \widehat{RQL}$

$\widehat{RPK} = 90° + \frac{\hat{C}}{2} = \widehat{RQL}$ (exterior angles for ΔPKC and ΔQLC).

$\Rightarrow$ Remains to prove: $|RP|\cdot|PK| = |RQ|\cdot|QL|$, or equivalently, $\boxed{|RP|\cdot|PC| = |RQ|\cdot|QC|}$ *

In $\Delta PKB \equiv \Delta PKC$: $\frac{|PK|}{|PB|} = \frac{|PK|}{|PC|} = \sin \widehat{PBK} = \sin \widehat{PCK} = \sin \frac{\hat{C}}{2}$

In $\Delta QLA \equiv \Delta QLC$: $\frac{|QL|}{|AQ|} = \frac{|QL|}{|QC|} = \sin \widehat{QAL} = \sin \widehat{QCL} = \sin \frac{\hat{C}}{2}$

$\Rightarrow \boxed{\frac{|PK|}{|PC|} = \frac{|QL|}{|QC|}}$

Note: $|RP|\cdot|PC|$ = power of point P = radius$^2 - |OP|^2$

$|RQ|\cdot|QC|$ = power of point Q = radius$^2 - |OQ|^2$

where O = circumcentre, radius = circumradius

Remains to prove: $\boxed{|OP| = |OQ|}$.

$\widehat{OQP} = 90° - \frac{\hat{C}}{2}$, $\widehat{OPQ} = \widehat{KPC} = 90° - \frac{\hat{C}}{2} = \widehat{OQP} \Rightarrow \Delta OQP$ isosceles $\Rightarrow$

$\Rightarrow |OP| = |OQ|$ q.e.d.

Solution II:

Equal angles:

$PK = \perp$ bisector of $BC \Rightarrow |PC| = |PB| \Rightarrow \widehat{PCB} = \widehat{PBC}$.

$QL = \perp$ bisector of $AC \Rightarrow |PC| = |PA| \Rightarrow \widehat{PCA} = \widehat{PAC}$

$\widehat{RAB} = \frac{\overset{\frown}{RB}}{2} = \widehat{RCB} = \frac{\hat{C}}{2} = \widehat{RCA} = \frac{\overset{\frown}{RA}}{2} = \widehat{RBA}$

similar Δ-s: $\Rightarrow \Delta RAB \sim \Delta QAC \sim \Delta PBC$ isosceles.

"Rotated Δ-s": $\Delta RAB \sim \Delta QAC \Rightarrow \frac{|RA|}{|RB|} = \frac{|AQ|}{|AC|}$ and $\widehat{RAQ} = \widehat{BAC} \Rightarrow \Delta ARQ \sim \Delta ABC$

$\Rightarrow \Delta RBP \sim \Delta ABC$ $\Rightarrow$

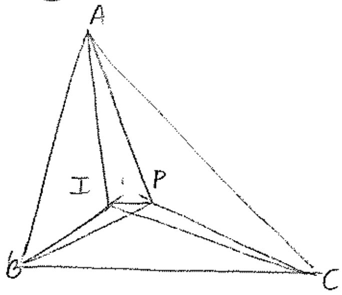
similarly: $\Delta RBA \sim \Delta PBC \Rightarrow$

$\Rightarrow \Delta ARQ \sim \Delta RBP \Rightarrow 1 = \frac{|AR|}{|RB|} = \frac{|RQ|}{|BP|} = \frac{|AQ|}{|RP|}$

$\Rightarrow |RQ| = |BP|$ and $|AQ| = |RP|$

$\Rightarrow |RP|\cdot|PB| = |RQ|\cdot|AQ| \Rightarrow$ *.

IMO 2006 Problem 4



We know:

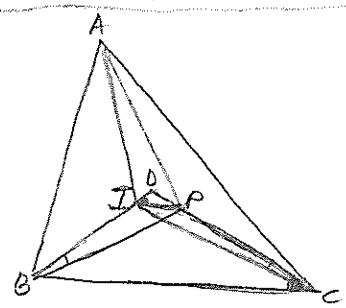
- I = incentre of ΔABC
- $\widehat{PBA} + \widehat{PCA} = \widehat{PBC} + \widehat{PCB}$. $(*)$

To prove: $|AP| \geq |AI|$ with "=" iff $P = I$.

Strategy: Given the data we have to work with, clearly we're to use an angle based strategy.
Rewrite $|AP| \geq |AI|$ as an inequality among angles in ΔAPI.
Simplify $(*)$ inasmuch as possible using the angles of ΔABC as reference.

Hint ① Can you find a cyclic quadrilateral?
What other conclusions about angles can you draw from here?
(You may extend and intersect BI and CP).

IMO 2006 Problem 4



We know:

I = incentre of $\triangle ABC$

$\widehat{PBA} + \widehat{PCA} = \widehat{PBC} + \widehat{PCB}$

To prove: $|AP| \geq |AI|$,

"=" iff $P = I$.

Solution: $\widehat{PBA} + \widehat{PCA} = \widehat{PBC} + \widehat{PCB} \iff \boxed{\widehat{PBA} - \widehat{PBC} = \widehat{PCB} - \widehat{PCA}}$.

Case ⓪ If $\widehat{PBA} = \widehat{PBC}$ then $\widehat{PCB} = \widehat{PCA} \Rightarrow P = I$

① If $\widehat{PBA} > \widehat{PBC}$ then $\widehat{PCB} > \widehat{PCA} \Rightarrow$ line PB is inside $\widehat{IBC}$, line PC is inside $\widehat{ACI}$

$\Rightarrow P$ is inside $\triangle AIC$.

Compare to $\hat{B}, \hat{C}$:

$$\left.\begin{array}{l} \widehat{PBA} = \frac{\hat{B}}{2} + \widehat{PBI} \\ \widehat{PBC} = \frac{\hat{B}}{2} - \widehat{PBI} \end{array}\right\} \Rightarrow \boxed{\widehat{PBA} - \widehat{PBC} = 2\widehat{PBI}}$$

similarly, $\boxed{\widehat{PCB} - \widehat{PCA} = 2\widehat{PCI}}$

$\Rightarrow \widehat{PBI} = \widehat{PCI}$

$\Rightarrow IPCB$ cyclic

Let BI intersect CP at $D \Rightarrow$ inside $\triangle AIP$.

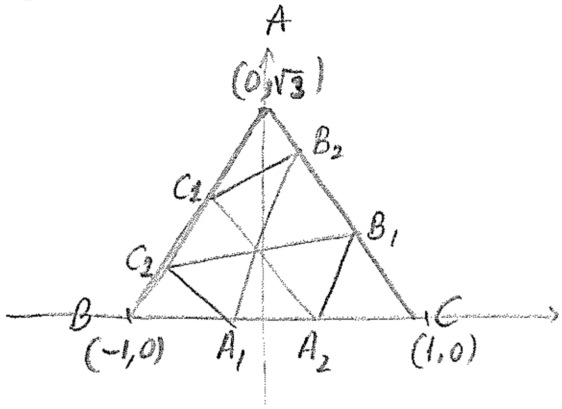
$$\left.\begin{array}{l} \widehat{AIP} = \widehat{AID} + \widehat{DIP} \\ \widehat{AID} = \frac{\hat{A}}{2} + \frac{\hat{B}}{2} \text{ (exterior angle to } \triangle ABI) \\ \widehat{DIP} = \widehat{PCB} \text{ (because } IPCB \text{ cyclic)} > \frac{\hat{C}}{2} \end{array}\right\} \Rightarrow \widehat{AIP} > \frac{\hat{A}}{2} + \frac{\hat{B}}{2} + \frac{\hat{C}}{2} = 90^\circ$$

$\Rightarrow \widehat{AIP} > \widehat{API}$

$\Rightarrow \sin\widehat{AIP} > \sin\widehat{API} \Rightarrow |AP| > |AI|$

$$\left(\begin{array}{l} \text{because } \dfrac{|AP|}{\sin\widehat{AIP}} = \dfrac{|AI|}{\sin\widehat{API}} \\ \Rightarrow |AP| = \dfrac{\sin\widehat{AIP}}{\sin\widehat{API}} \cdot |AI| > |AI| \end{array}\right)$$

IMO 2005 Problem 1



We know:

- ΔABC equilateral.
- $|A_1A_2| = |A_2B_1| = |B_1B_2| = |B_2C_1| = |C_1C_2| = |C_2A_1|$.

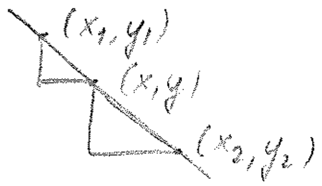
To prove:

A_1B_2, A_2C_1, B_1C_2 collinear

Strategy: This problem is suitable for a coordinate-based approach because:

- equilateral triangle $\Rightarrow$ coordinates easy to choose/calculate
- to prove intersection of lines $\Leftrightarrow$ a system of linear equations has solutions.

Basic tools:



- Equation of line through 2 points (x_1,y_1) and (x_2,y_2):

$$\boxed{\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1}} \Leftrightarrow \boxed{(x_2-x_1)\cdot y = (y_2-y_1)\cdot x + (x_2y_1 - x_1y_2)}$$

- 3 linear equations

$$\begin{cases} a_1 y = b_1 x + c_1 \\ a_2 y = b_2 x + c_2 \\ a_3 y = b_3 x + c_3 \end{cases}$$

have common solution iff

(a_1, b_1, c_1) can be written as a linear combination of (a_2, b_2, c_2) and (a_3, b_3, c_3).

(example: $a_1 = a_2 + a_3$, $b_1 = b_2 + b_3$, $c_1 = c_2 + c_3$).

IMO 2005 Problem 1 Solution Sketch:

- $A_1(x_1, 0)$, $A_2(x_2, 0)$
- $B_1(s_1, t_1)$, $B_2(s_2, t_2)$ are on the line AC of equation:

$$\boxed{y = -\sqrt{3}\,x + \sqrt{3} = \sqrt{3}(1-x)}$$

$$\Rightarrow B_1(s_1, \sqrt{3}(1-s_1)) \quad B_2(s_2, \sqrt{3}(1-s_2)).$$

- $C_1(u_1, v_1)$, $C_2(u_2, v_2)$ are on the line AB of equation:

$$\boxed{y = \sqrt{3}(x+1)}$$

$$\Rightarrow C_1(u_1, \sqrt{3}(u_1+1)), \quad C_2(u_2, \sqrt{3}(u_2+1)).$$

$$|A_1A_2| = |A_2B_1| = |B_1B_2| = |B_2C_1| = |C_1C_2| = |C_2A_1|$$

$$\left.\begin{aligned} \Rightarrow \underline{(x_2-x_1)^2} &= (s_1-x_2)^2 + 3(1-s_1)^2 = \underline{4(s_1-s_2)^2} = (s_1-u_1)^2 + 3(s_1+u_1)^2 = \\ &= \underline{4(u_1-u_2)^2} = (u_2-x_1)^2 + 3(u_2+1)^2. \end{aligned}\right\}(*)$$

$$\Rightarrow \begin{cases} s_2 = s_1 - \dfrac{x_2-x_1}{2} & (**) \\ u_1 = u_2 + \dfrac{x_2-x_1}{2} & \ldots \end{cases}$$

Equation of A_2C_1:

① $(u_1 - x_2)\, y = x \cdot \sqrt{3}(u_1+1) - \sqrt{3}\, x_2 (u_1+1)$.

Equation of A_1B_2

② $(s_2 - x_1)\, y = x \cdot \sqrt{3}(1-s_2) - \sqrt{3}\, x_1 (1-s_2)$

Equation of B_1C_2

③ $(u_2 - s_1) \cdot y = x \cdot \sqrt{3}(u_2+s_1) + \sqrt{3}u_2(1-s_1) - \sqrt{3}(u_2+1)s_1$

---

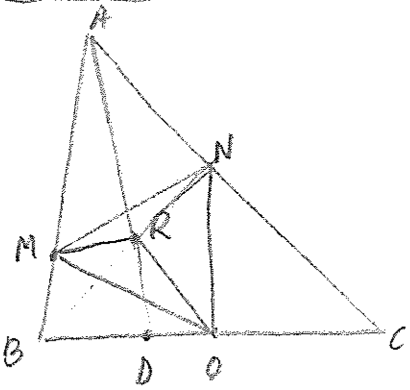
①-②-③: $\underbrace{(u_1 - x_2 - s_2 + x_1 - u_2 + s_1)}_{0 \text{ by } **} \cdot y = x \cdot \sqrt{3}\,\underbrace{(u_1 + 1 - 1 + s_2 - u_2 + s_1)}_{0 \text{ by } **} - \sqrt{3} \cdot U$

where $U = x_2 u_1 + \underline{x_2 - x_1} + x_1 s_2 - u_2 + u_2 s_1 + u_2 s_1 + s_1$

Using $\begin{matrix}*\\ **\end{matrix}\Big\}$ eventually $U = 0$.

thus if x, y satisfy ②, ③ $\Rightarrow$ they also satisfy ①.

IMO 2004 Problem 1



We know: $|AB| \neq |AC|$.

- O = midpoint of BC
- O = circumcentre for $MNCB$
- AR = ∢ bisector of $\widehat{BAC}$
- OR = ∢ bisector of $\widehat{MON}$

To prove:

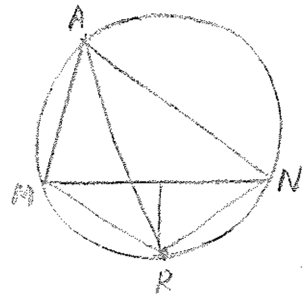
$\mathscr{C}(BMR)$ intersects $\mathscr{C}(CNR)$ at point on D.
(circumcircle)

Strategy: Cyclic quadrilaterals.

Let $MRDB$ cyclic. Then $RNCD$ cyclic $\Leftrightarrow$ $MANR$ cyclic (angle chasing)

But $|OM| = |ON|$
OR angle bisector of $\widehat{MON}$ $\Big\} \Rightarrow$ OR = ⊥ bisector of MN
AR = angle bisector of $\widehat{MAN}$.

Lemma



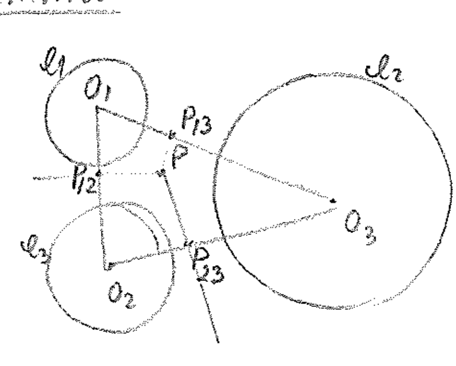
∢ bisector of $\widehat{A}$ and ⊥ bisector of MN intersect on circumcentre (at midpoint of $\widehat{MN}$).

☺ Bonus points: Prove D is on the line AR.

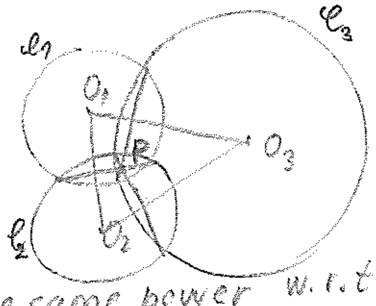
Hint: The common chord of $\mathscr{C}(BMR)$ and $\mathscr{C}(ABC)$ is AB
common chord of $\mathscr{C}(CNR)$ and $\mathscr{C}(ABC)$ is AC
$\mathscr{C}(BMR)$ and $\mathscr{C}(CNR)$ is RD $\Big\} \Rightarrow$

Lemma $\Rightarrow$ AB, AC, RD intersect at A.

Lemma



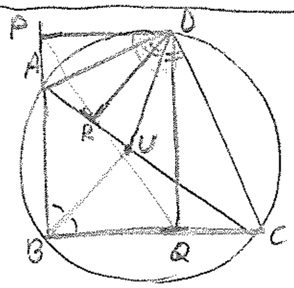
or



$\exists P$ having the same power w.r.t. $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3$.
If the circles have common chords, then P is at the intersection of the 3 common chords.

Hint: $|P_{ij}O_i|^2 - |P_{ij}O_j|^2 = r_i^2 - r_j^2$. Use Pythagoras' to prove ⊥ from P_{ij}-s intersect at P.

IMO 2003 Problem 4



We know:

$DP \perp AB$; $DQ \perp BC$, $DR \perp CA$.

BU = ∡ bisector of $\widehat{ABC}$

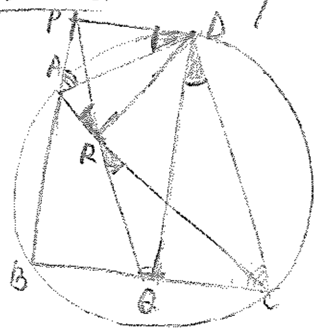
DU = ∡ bisector of $\widehat{ADC}$

To prove:

$|PR| = |RQ|$.

Strategy: The data are all about angles, but we have to prove an equality of segments. We cannot use $\widehat{RPQ} = \widehat{RQP}$ because P, R, Q are collinear:

Lemma Simpson's line:



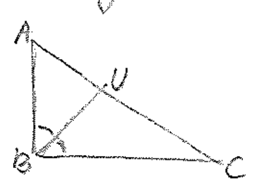
$\left.\begin{array}{l} DP \perp AB,\ DQ \perp BC,\ DR \perp CA \\ D, A, B, C \text{ concyclic} \end{array}\right\} \Rightarrow P, Q, R$ collinear.

(Proof: $\widehat{ARP} = \widehat{ADP} = \widehat{CDQ} = \widehat{QRC}$ because:

$\left\{\begin{array}{l} ARDP \text{ cyclic } (\hat{R} = \hat{P} = 90°) \\ RDCQ \text{ cyclic } (\widehat{DRC} = \widehat{DQC} = 90°) \\ \Delta DPA \sim \Delta DQC \quad : \quad \hat{P} = \hat{Q} = 90° \\ \qquad\qquad\qquad\qquad\quad \widehat{DAP} = \widehat{DCQ} \text{ (ABCD cyclic)} \end{array}\right.$)

Return to problem ⇒ Strategy: Relate angles with segments by

Lemma on ∡ bisectors:

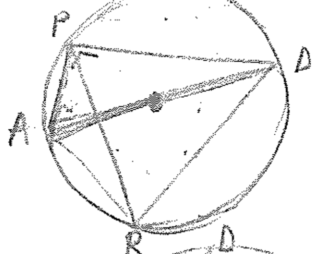


$\frac{|AB|}{|BC|} = \frac{|AU|}{|UC|}$.

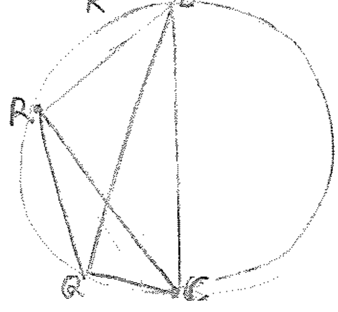
And similarly for DU.

sines theorem:

Hint:



$\frac{|PR|}{\sin \widehat{PDR}} = 2 \text{ radius} = |AD|$



$\frac{|RQ|}{\sin \widehat{RDQ}} = |DC|$

Similarly, apply sines' theorem in ΔABC.

(Relate $\widehat{PDR}$, $\widehat{RDQ}$ to angles in ΔABC.)